COMS E6998: Proof Complexity and Applications (Spring '25) March 6, 2025

# Lecture 6: Bounded-depth Frege

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## 1 Overview

Today's focus is on proving bounded-depth Frege size lower bounds for the *Pigeon Hole Principle*. First, we prove the size lower bounds for computing the *Parity* function. Building on this, we then prove the size lower bounds for *PHP*. Formally speaking, our goal is to show the following:

**Theorem 1.** *Any depth-d Frege proof of* $PHP_n^{n+1}$ *requires size* $2^{n^{\epsilon_d}}$ *with* $\epsilon_d = \frac{1}{6^d}$.

The result we show today is originally due to [Ajt94] which showed a superpolynomial bound, and it was then improved by [PBI93, KPW95] to an exponential lower bound. The bound was then further strengthened by [Hås23].

## 2 Decision Trees and Restrictions

Before diving into the main theorem, we first introduce some essential definitions.

**Definition 2.** *Let decision tree* $T$ *over* $x_1 \ldots x_n$ *to be a binary tree where each internal node is labeled with a variable* $x_i$*; the two out edges of an internal vertex are labeled with 0, 1 respectively, and each leaf is labeled by either 0 or 1. An out edge labeled 0 means that this edge corresponds to setting* $x_i = 0$ *and the other edge labeled 1 corresponds to setting* $x_i = 1$*. Thus, each path in* $T$ *corresponds to a partial assignment* $\sigma$*, which assigns binary values to all variables queried on the path from the root of* $T$ *to a leaf.*

*A decision tree* $T$ *over* $x_1...x_n$ *represents a DNF* $f$ *if all paths in* $T$ *with associated partial matching restriction* $\sigma$, $f|_\sigma$ *= leaf value of path* $\sigma$.

The following in Fig. 1 is an example decision tree over $x_1...x_4$:

**Definition 3.** *t-DNF is a disjunction of terms, where each term has a maximum size of t.*

**Definition 4.** *Let* $f$ *be a t-DNF, then a restriction* $\rho : \{x_1...x_n\} \to \{0, 1, *\}$ *is a partial assignment that sets the underlying variables to 0, 1, or* $*$*. We apply the restriction* $\rho$ *term by term, if any term evaluates to 1 then* $f_\rho = 1$*, otherwise* $f_\rho$ *can be viewed as a DNF consists of the original terms of* $f$ *but each term only has the unassigned literals while the assigned literals are removed.*

**Definition 5.** *Let* $f$ *be a t-DNF and* $\rho$ *a partial restriction. The canonical decision tree for* $f$ *restricted by* $\rho$ *is defined as follows:*

- *If* $f|_\rho$*=0,* $T(f|_\rho)$ *consists of a single node labeled* 0.
- *If* $f|_\rho$*=1,* $T(f|_\rho)$ *consists of a single node labeled* 1.

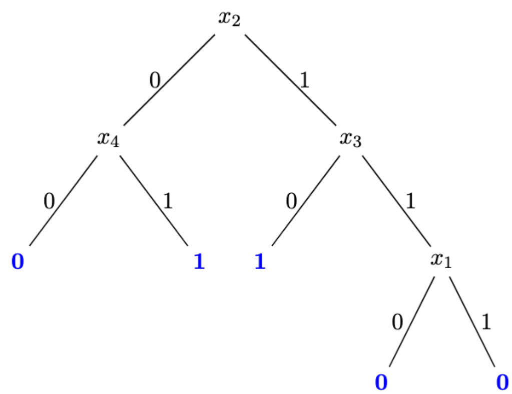


Figure 1: An example decision tree with variables $x_1, x_2, x_3, x_4$

- *Suppose* $f|_\rho = C_i \vee C_{i+1} \vee ...C_k$*. Then,* $T(f|_\rho)$ *first query all free literals of* $C_i$*. Each path from the root to a leaf represents a partial assignment* $\sigma_i$*.*
  - *If* $C_i|_{\rho\cup\sigma_i} = 1$*, then the corresponding leaf is labeled 1.*
  - *If* $C_i|_{\rho\cup\sigma_i} = 0$*, then we recursively construct its subtree* $T'|_{\rho\cup\sigma_i}$ *with* $f'|_{\rho\cup\sigma_i} = C_{i+1} \vee ...C_k$*.*

The following in Fig. 2 is an example canonical decision tree $T$ with f=$x_1\bar{x_2} \bigvee x_4$.

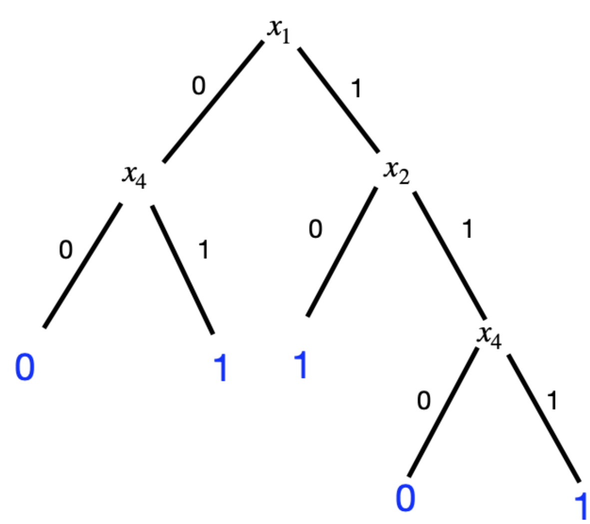


Figure 2: An example canonical decision tree for f=$x_1\bar{x_2} \bigvee x_4$

**Definition 6.** *Random Restrictions* $\mathbb{P}_n^p$*: set of restrictions* $\rho$ *on domain* $x_1...x_n$ *such that for each* $x_i$ *we set* $x_i$ *to 0 w.p.* $\frac{(1-p)}{2}$*, 1 w.p.* $\frac{(1-p)}{2}$*, and leave it unset (set to *) w.p.* $p$*.*

## 3 Lower Bounds for Parity

In order to prove the lower bounds for bounded-depth Frege on $PHP$, we first find the lower bounds for *Parity*. Here, we show the following result due to [Has86]:

**Theorem 7.** *Parity requires* $2^{n^{\epsilon_d}}$ *size* $AC^0_d$*-circuit,* $\epsilon_d \approx \frac{1}{2d}$

On a high level, our proof proceeds as follows:

- Assume for contradiction $C$ is an $AC^0_d$-circuit of polynomial size computing $Parity$ over $x_1...x_n$.
- Repeatedly apply restrictions $\rho_1...\rho_d$ to shrink $C$ into circuits $C_1 = C|_{\rho_1}$ (depth = d-1), $C_2 = C|_{\rho_1\rho_2}$ (depth = d-2) ... $C_d = C|_{\rho_1\rho_2...\rho_{d-1}}$. In the end, $C_d$ is a trivial circuit that cannot compute $Parity$ on the remaining unset variables, hence forming a contradiction.

Above shows that our proof relies on finding restrictions that successfully shrink circuit $C$, and the switching lemma guarantees the existence of such restrictions.

**Lemma 8.** *Switching Lemma [Has86]: Let $f$ be a r-DNF over $x_1...x_n$ and $p \leq \frac{1}{4}$, then* $\Pr_{\rho\in\mathbb{P}^p_n}[T(f|_\rho)$ *has depth* $\geq s] \ \leq (4pr)^s$ .

*Proof of Theorem 7.* Lets proceed with proof by contradiction. Assume C is a size $\varsigma$ $AC^0[d]$ circuit for $Parity_n$, $\varsigma < \frac{1}{2d(4pr)^s}$, $p = 1/8r$, and $r = s = n^{1/2d}$. Without loss of generality, suppose bottom level of C consists of $m$ rDNFs. Since C has size at most $\varsigma$, then C has at most $\varsigma$ rDNFs at the bottom. When converting a r-DNF into a decision tree, the switching lemma says that for a random restriction $\rho_i \sim \mathbb{P}^p_n$:

$$Pr_{\rho_i}[T(f|\rho_1) \ has \ depth > s] < (4pr)^s$$

Through union bound we see that for any random restriction $\rho_i \sim \mathbb{P}^p_n$:

$$Pr_{\rho_i}[\exists j \in [m], T(f_j|\rho_1) \ has \ depth > s] < m(4pr)^s < \varsigma(4pr)^s < 1/2d$$

Above shows there exists a random restriction $\rho_1$ such that it converts all bottom level rDNFs $f_i|\rho_1$ to decision trees $T_i(f|_{\rho_1})$ with depth at most $s$. For each $T_i(f|_{\rho_1})$, if we look at its 1-leaves and the path $\pi_t$ from root to the leaf, we see that $f_i|_{\rho_1}(x) = \bigvee_{t \ labeled \ 1} \bigwedge_{x_j \ queried \ in \ \pi_t} x_i^{b_i}$, with $x_i^{b_i} = x_i$ or $\neg x_i$. This shows $f_i$ can be expressed as a sDNF, since there are at most s variables queried for each path $\pi_t$. Similarly, looking at the 0-leaves of $T_i(f|_{\rho_1})$ we see that $\neg f_i|_{\rho_1}$ can be expressed as a sDNF. Applying DeMorgan to $\neg f_i|_{\rho_1}$ and its associated sDNF formula, we get that $f_i|_{\rho_1}$ can be expressed as a sCNF. This shows each $f_i|_{\rho_1}$ can be expressed as both sDNF and sCNF.

Take a bottom level $f_i$, if the parent gate is $AND$, then under restriction $\rho_1$ we convert $f_i|_{\rho_i}$ to a sCNF. Then, we can merge a layer of this sCNF with the parent $AND$ gate. Similarly, if the parent gate is $OR$, then we convert $f_i|_{\rho_i}$ to a sDNF and merge with the parent $OR$ gate. This process reduces the depth of C by 1. Repeating the above procedure $d$ times, we get $\rho_1...\rho_d$ and $C|_{\rho_1...\rho_d}$ is a depth $\leq r$ decision tree for $Parity_{n'}$, where $n'$ is the number of unset variables after $\rho_1...\rho_d$. As long as $n' > r$, the above shows that $Parity_{n'}$ can be decided by a decision tree with depth $< n'$, which forms a contradiction.

Above, we see for a random restriction in any round $\rho_i$ it fails with probability $< 1/2d$ to reduce the depth of $C$ by 1. Using union bound, we see that the probability for any random restriction to fail in some round $< (d-1)/2d < 1$. Hence, there exists a sequence of restrictions $\rho_1 ... \rho_{d-1}$ that successfully reduces $C$ to a trivial circuit.

Concluding the above, we see that $\varsigma \geq \frac{1}{2d(4pr)^s}$, which after some calculation gives $\varsigma \geq 2^{n^{\epsilon_d}}$.

□

## 4 Lower Bounds for PHP

Recall an $AC_d^0$ Frege proof of $PHP_n^{n+1}$ is a sequence of $AC_d^0$ formulas $F_1, F_2, ... F_m$ such that each $F_i$ is either an axiom, or follows from one or two previous lines through a valid Frege rule. In the end, $F_m = PHP_n^{n+1}$.

Here is an naive attempt at finding lower bounds to PHP based on our proof for Theorem 7. Assume for contradiction that we have a size $\varsigma$ $AC_d^0$ Frege proof $\Pi$ for $PHP_n^{n+1}$, and we try applying a sequence of restrictions to reduce the depth of $\Pi$. Eventually, the depth of $\Pi$ becomes 1, and this gives a contradiction as there exists no $AC_d^0$ Frege proof for $PHP_n^{n+1}$ with depth=1.

However, a problem with our attempt is that every line in $\Pi$ is a tautology, and hence each line is already equivalent to a depth=1 trivial formula. Therefore, we must find a way to differentiate between a complicated depth d formula that evaluates to true, and other trivial formulas that also evaluate to true. Note that if we think of n as infinite, then there exists a bijection between $\{1, 2, ... n+1\}$ and $\{1, 2, ... n\}$. Building on this idea, we define a family of partial restrictions that not only satisfies the restrictions of $PHP_n^{n+1}$ but also enables depth reduction.

Here, we formally introduce the size lower bounds for $PHP$.

**Theorem 9.** *PHP requires $2^{n^{\epsilon_d}}$ size $AC_d^0$-circuit lower bounds, $\epsilon_d \approx \frac{1}{6^d}$*

Before presenting the full proof of Theorem 9, we first introduce some necessary elements.

**Definition 10.** *Matching restrictions $\rho$ over $\{P_{i,j}, i \in D, j \in R\}$ is a partial 1-1 mapping of size $n-g$ corresponding restriction. Here, $n = min(|D|, |R|)$ and $g$ is the number of variables left unset. Suppose $\rho$ maps $i$ to $j$, then*

- $P_{i,j} = 1$
- $P_{i,j'} = 0,\ \forall j' \neq j$
- $P_{i',j} = 0,\ \forall i' \neq i$

**Definition 11.** *A matching disjunction is an **OR** of matching terms, where a matching term corresponds to a partial one-to-one mapping. A r-disjunction is an **OR** of matching terms with size at most r. Below is an example of a 2-disjunction:*

$$P_{1,2}P_{3,4} \bigvee P_{3,2}P_{4,1} \bigvee P_{2,3}$$

**Definition 12.** *Let f be a r-disjunction, $\rho$ be a matching restriction, then the restriction of a r-disjunction $f|_\rho$ is defined as setting certain variables as matching with one another. For instance,*

$$P_{1,2}P_{3,4} \bigvee P_{3,2}P_{4,1} \bigvee P_{2,3}|_{1\to 3, 4\to 1} = P_{3,2}$$

**Definition 13.** *A matching decision tree over set $D \cup R$ is a rooted directed tree $T$ such that*

- *Internal nodes are labeled by elements of $D \cup R$.*
- *Leaves are labeled by 0 or 1.*
- *Suppose the root is labeled by $i \in D$, then for each $j \in R$ there is one edge from root labeled $i \to j$.*
- *Suppose the root is labeled by $j \in R$, then for each $i \in D$ there is one edge from root labeled $i \to j$.*
- *Take $T^{(i\to j)}$ to be the subtree such that its root is connected to the root of $T$ through an edge labeled $i \to j$. Then, $T^{(i\to j)}$ is a matching decision tree over $D' \cup V'$, with $D' = D - \{i\}$ and $V' = V - \{j\}$.*

Suppose $D = \{1, 2, 3, 4\}$ and R=$\{1', 2', 3'\}$. Then, the matching decision tree $T$ in Fig. 3 corresponds to the matching disjunction $P_{2,1'}P_{3,2'} \bigvee P_{2,3'}P_{4,1'}$.

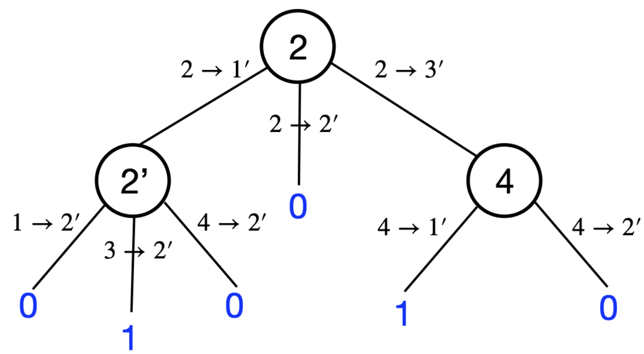


Figure 3: A matching decision tree for $P_{2,1'}P_{3,2'} \bigvee P_{2,3'}P_{4,1'}$

**Definition 14.** *Let $\rho$ be a matching restriction and $T$ a matching decision tree. We define $T|_\rho$ inductively:*

1. *If $T$ consists of a single node $T|_\rho = T$.*
2. *If $T$ consists of at least 2 nodes and suppose the root of $T$ is labeled with vertex $i$, then*
   - *If $\rho$ maps $i \to j$ for some $j$, then $T|_\rho = T'_\rho$ with $T'$ being the subtree of root labeled with $i \to j$.*
   - *If $\rho$ does not map any element to or from $i$, then $T|_\rho$ has root labelled by $i$ with subtrees $T'|_\rho$ where $T'$ is connected to the root by an edge $i \to k$ where $\rho$ does not fix $k$.*

Suppose $D = \{1, 2, 3, 4\}$ and R=$\{1', 2', 3'\}$. Then, the matching decision tree $T$ in Fig. 4 corresponds to the matching disjunction $P_{2,1'}P_{3,2'} \bigvee P_{2,3'}P_{4,1'}$ after applying restriction $2 \to 1'$.

**Definition 15.** *Let $T$ be a matching decision tree over variables of $PHP_n^{n+1}$ with depth $< n$, and let $f$ be a matching disjunction. $T$ **represents** $f$ if for all paths in $T$ with associated partial matching restriction $\sigma$, $f|_\sigma$ = leaf value of path $\sigma$.*

**Lemma 16.** *Let $T$ be a matching decision tree and $\rho$ a matching restriction. Then we have the following:*

1. *$Disj(T) \equiv Disj(T|_\rho)$*

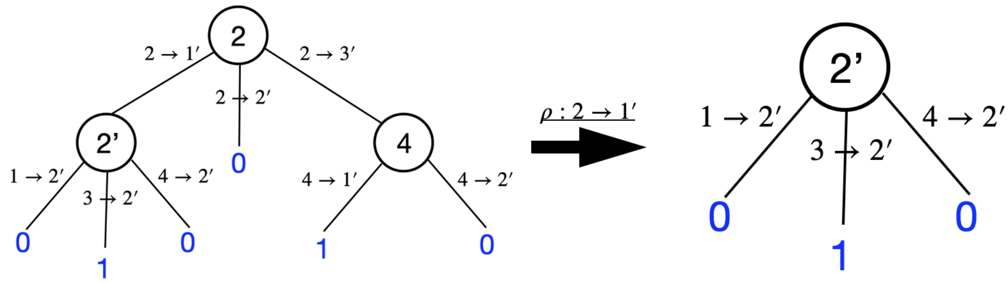


Figure 4: The matching decision tree of $P_{2,1'}P_{3,2'} \bigvee P_{2,3'}P_{4,1'}$ with the restriction $2 \to 1'$

2. *If $T$ is complete over $D^{n+1} \cup R^n$, then $T|_\rho$ is complete over $D^{n+1}|_\rho$ and $R^n|_\rho$. Here,* ***complete*** *means that every assignment where all holes are assigned pigeons is associated with an unique leaf.*

3. *$(T|_\rho)^c = T^c|_\rho$, with the complement being the toggling of all values of leaves.*

4. *If $l$ is a leaf in $T|_\rho$, then there exists a leaf $l'$ in $T$ with same label as $l$ so that $\Pi(l') \subseteq \Pi(l) \cup \rho$ (where $\Pi(l')$ is the partial matching associated with path in $T$ from root to $l'$).*

5. *If $T$ represents the matching disjunction $f$, then $T|_\rho$ represents the matching disjunction $f|_\rho$.*

**Definition 17.** *Let $f$ be an $r$-disjunction and $\rho \in Q_n^p$ a matching restriction. The canonical matching decision tree for $f$, $T(f)$, is defined as follows:*

- *If $f = 0$: $T(f|_\rho)$ consists of a single node labeled 0.*
- *If $f = 1$: $T(f|_\rho)$ consists of a single node labeled 1.*
- *Else: Let $t_1$ be the first matching term of $f$. Create the complete matching decision tree over the set $s' \subseteq D \cup R$ of pigeons and holes mentioned in the first clause $C_1$. If $t_1$ is forced into a value of 0 or 1, then the process can be terminated early. First, each leaf $i$ is associated with a matching restriction $\sigma_i$. Then, we inductively replace each leaf $i$ with the canonical matching decision tree $T(f|_{\sigma_i})$.*

Fig. 5 is an example of the canonical matching decision tree $T$ for: $D = \{1,2,3,4\}$, $R = \{1',2',3'\}$, $f = P_{21'} \cup P_{23'}P_{42'}$. Let $t_1 = P_{21'}$ and $t_2 = P_{23'}P_{42'}$. In $T$, the protocol first queries the pigeons/holes mentioned in $t_1$, then the pigeons/holes mentioned in $t_2$.

**Definition 18.** *Random Restrictions $Q_n^p$: set of all matching restrictions $\rho$ over $D = [n+1]$ and $R = [n]$ such that after applying $\rho$, there are still $pn+1$ pigeons and $pn$ holes unset.*

**Lemma 19.** *PHP Switching Lemma: Let $f$ be a $r$-disjunction, then $Pr_{\rho \in Q_n^p}[T(f|_\rho)$ has depth $\geq s]$ $\leq (11p^4n^3r)^s$.*

Here, we note that though the $PHP$ switching lemma provides a larger upper bound than the $Parity$ switching lemma, we will see shortly that this is still sufficient to prove our intended size lower bound for $PHP$. Our proof of Theorem 8 will utilize the follow Switching Lemma procedure that reduces the depth of a formula $L$:

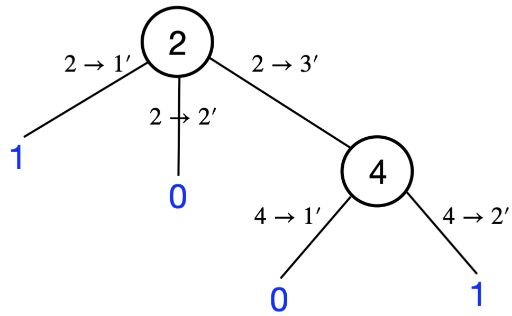


Figure 5: The canonical matching decision tree for $f = P_{21'} \cup P_{23'} P_{42'}$, with the protocol first querying the pigeons/holes mentioned in the first clause, and then the pigeons/holes mentioned in the second clause

1. Without loss of generality: $L$ is a depth $d$ formula over basis $\bigvee$ and $\neg$, and the bottom two layers are r-disjunctions $f_i$.
2. Let $\rho_1 \in Q_n^p$ be matching restriction such that $\forall i\ T(f_i|_{\rho_1})$ has depth $\leq r$ ($\rho_1$ guaranteed to exist by the $PHP$ Switching Lemma).
3. Define $L|_{\rho_1}$ such that for all bottom-level depth-3 subformulas $h_j$:
   - Convert $\neg f_i|_{\rho_1} \to \neg T(f_i|_{\rho_1}) \to T^c(f_i|_{\rho_1})$
   - Convert $\bigvee_{i=1}^q (\neg f_i|_{\rho_1}) \to \bigvee_{i=1}^q \bigvee_{t \text{ a 1-path in } T(f_i|_{\rho_1})} t$ .

After the application of *Step 3*, $L|_{\rho_1}$ is converted into a new formula of $OR$ with depth $d-2$, while the leaves continue to be r-disjunctions. After repeating *Step 3* for $d-1$ times, we have $L|_{\rho_1 \dots \rho_{d-1}}$ which converts to a single matching decision tree of height $\leq r$. From here, we then obtain a contradiction by transforming the initial formula into a 1-tree and the final formula after restrictions into a 0-tree.

[To obtain the bounds as shown in Thm. 9, we set the parameters as follows: Let $n_0 = n$ and $n_i =$ number of unset holes after round $i$ of applying restrictions. As such, $n_{i+1} = p_{i+1} n_i$. Set $p_{i+1} = n_i^{\frac{-5}{6}}$ and $r_i = s_i = n_i^{\frac{1}{6}}$. Then, the $PHP$ switching lemma fails to find a satisfying restriction $\rho$ with probability $\leq 11(n_i^{\frac{-20}{6}} n_i^3 n_i^{\frac{1}{6}})^{r_i} = 11(n_i^{\frac{-20}{6}+\frac{18}{6}+\frac{1}{6}})^{r_i} = 11(n_i^{\frac{-1}{6}})^{r_i}$. For $n_i$ large enough, we have $11 n_i^{\frac{-1}{6}} < 1/2$, and therefore $11(n_i^{\frac{-1}{6}})^{r_i} < 1/2^{r_i}$. For base case, we achieve $n_{d-1} = n^{\frac{1}{6^{d-1}}}$ with $r_{d-1} = 1/6^d$. With this setting, we then obtain lower bound on size $\vartheta > 2^{n^{r_{d-1}}} \sim 2^{n^{1/6^d}}$.]

**Proof Sketch of Theorem 8**:

- Let $\Pi$ be an alleged size $< \varsigma$ $AC^0[d]$- Frege proof of $PHP_n^{n+1}$. $\Pi = \{L_1, ..., L_m\}$.
- Following the PHP switching lemma procedure sketched above (using parameters described) which can be applied iteratively $d-1$ times to obtain a good sequence of restrictions $\rho_1, ..., \rho_{d-1}$ such that under $\rho = \rho_1 ... \rho_{d-1}$, we can convert $\Pi = \{L_1, ..., L_m\}$ into another sequence of formulas $\Pi^* = \{L_1^*, ..., L_m^*\}$ where $L_i^*$ are depth $\leq r$ matching decision trees, obtained by successively applying $\rho_1, ..., \rho_{d-1}$ to yield $L_i^1, ..., L_i^d - 1 = L_i^*$.

- To finish the lower bound, we need to reach a contradiction by showing that if the proof Π is sound, then $\Pi^*$ is also a "locally" sound proof of $PHP_n^{n+1}$ on the remaining $n'+1$ unset pigeons, and n' unset holes.

**Definition 20.** *k-evaluation: Let* $\Gamma$ *be a set of formulas closed under subformulas, over* $D^{n'+1} \cup R^{n'}$. *A k-evaluation for* $\Gamma$ *is an assignment of complete matching decision trees* $\mathcal{T}(A)$ *to all subformulas* $A$ *that occur in* $\Gamma$ *such that:*

1. $\mathcal{T}(A)$ *has depth* $\leq k$ *for all* $A$
2. - $\mathcal{T}(1)$ *is the tree with a single node labeled 1*
   - $\mathcal{T}(0)$ *is the tree with a single node labeled 0*
3. $\mathcal{T}(P_{ij})$ *is the full matching tree over* $D^{n'+1} \cup R^{n'}$ *with leaf* $l$ *labeled 1 if* $\Pi(l)$ *contains* $\{i,j\}$, *and 0 o.w.*
4. *If* $A$ *is a matching disjunction then* $\mathcal{T}(A)$ *represents* $\bigvee_i Disj(\mathcal{T}(A_i))$

**Lemma 21.** *Obtaining a k-evaluation: Let* $\Pi$ *be a size* $\varsigma$, *depth d Frege proof of* $PHP_n^{n+1}$. *Let* $\rho_1, ..., \rho_{d-1}$ *be the good restrictions guaranteed to exist by the PHP switching Lemma. Then:*
*There exists a k-evaluation for* $\Gamma = \{$ *all subformulas occurring in* $\Pi|_{\rho_1 \ldots \rho_{d-1}}\}$ *over* $D^{n'+1} \cup R^{n'}$.

[Proof of lemma omitted, proof idea: Prove inductively on depth d. Let $F_i$ = all subformulas occurring in Π that have depth $\leq i$. Then after stage $i$, we have a $k$- evaluation for the formulas $F_i$-$\rho_1, ..., \rho_{d-1}$.

Now we will now finish our proof of Theorem 8. Base case: Let $\Pi = \{L_1, .., L_m\}$ be the alleged proof of $PHP_n^{n+1}$. It is left to argue that if we have a $k$-evaluation $\mathcal{T}$ for all subformulas of $\Pi|\rho$ over $D^{n'+1} \cup R^{n'}$ where $k << n'$ then we reach a contradiction as follows:

- (A) on the one hand we can show that all axioms of $\Pi|_\rho$ convert to all-1 trees and the Frege rules preserve 1-trees, so every formula in $\Pi_\rho$ converts to a 1-tree.
- (B) On the other hand, the last line $L_m|_\rho$ converts to an all 0-tree.

We will now show why (B) holds. (B) states that for all $n > 1$, $PHP_n^{n+1}$ converts to a 0-tree under $\mathcal{T}$. $PHP_n^{n+1}$ consists of the disjunction of the following formulas

1. $\neg(\neg P_{i,k} \vee \neg P_{j,k})\ \ \forall i \neq j \leq n+1, k \leq n$
2. $\neg(P_{i,1} \vee P_{i,2} \vee ... \vee P_{i,n})\ \ \forall i \leq n+1.$

$\mathcal{T}$(Type (1) clauses) are all 0-trees: Since $\mathcal{T}(\neg(\neg P_{i,k} \vee \neg P_{j,k})) = \mathcal{T}^c(\neg P_{i,k} \vee \neg P_{j,k})$ (tree for $\neg P_{i,k} \vee \neg P_{j,k}$ with all leaf values toggled), to show $\mathcal{T}(\neg(\neg P_{i,k} \vee \neg P_{j,k}))$ is a 0-tree it suffices to show that $\mathcal{T}(\neg P_{i,k} \vee \neg P_{j,k})$ is a 1-tree. To accomplish this, note that $\mathcal{T}(\neg P_{i,k} \vee \neg P_{j,k}) = \mathcal{T}(Disj(\mathcal{T}^c(P_{i,k})) \vee \mathcal{T}(Disj(\mathcal{T}^c(P_{i,k})))$. For concreteness, consider $\mathcal{T}(\neg P_{1,3'} \vee \neg P_{2,3'})$.

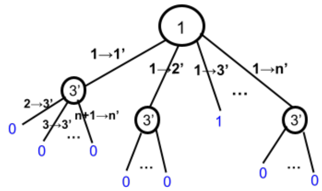


Figure 6: The matching decision tree $\mathcal{T}(P_{1,3'})$. Note that there is exactly one 1-leaf corresponding to the path $1 \to 3'$.

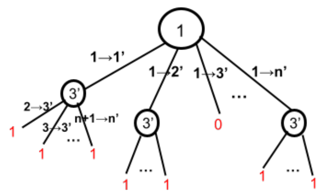


Figure 7: The matching decision tree $\mathcal{T}(\neg P_{1,3'})$. Note that there is exactly one 0-leaf corresponding to the path $1 \to 3'$.

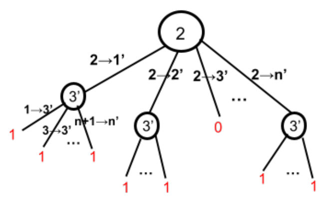


Figure 8: The matching decision tree $\mathcal{T}(\neg P_{2,3'})$. Note that there is exactly one 0-leaf corresponding to the path $2 \to 3'$.

Then $\mathcal{T}(\neg P_{1,3'} \vee \neg P_{2,3'}) = \mathcal{T}((\bigvee_{\text{all 1-paths } \sigma \text{ in } \mathcal{T}(\neg P_{1,3'}))} m_\sigma) \vee (\bigvee_{\text{all 1-paths } \sigma' \text{ in } \mathcal{T}(\neg P_{2,3'}}} m_{\sigma'}))$. This tree queries pigeons 1,2 and hole 3'. All the leaves of this tree are labeled with 1 since no partial matching over $\{2, 1, 3'\}$ maps $2 \to 3'$ and $1 \to 3'$. Can similarly argue for all type (1) clauses.

Finally, to argue that tall type (2) clauses of the form $\mathcal{T}(\neg(P_{i,1} \vee P_{i,2} \vee ... \vee P_{i,n}))$ yield a 0-tree, it suffices to show that $\mathcal{T}((P_{i,1} \vee P_{i,2} \vee ... \vee P_{i,n}))$ is a 1-tree. Consider figure 9 below.

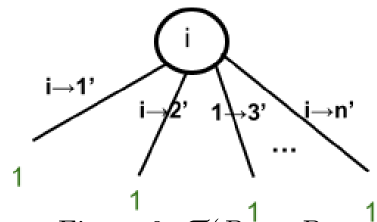


Figure 9: $\mathcal{T}(P_{i,1} \vee P_{i,2} \vee ... \vee P_{i,n})$

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