Today

I. Frege Systems : Several examples, Soundness + Completeness

$\mathcal{C}$-Frege systems

Soundness is "complete"

Normal-Form Theorems

II. Extended Frege : Defn

Soundness formalized combinatorially

Equivalent Systems

III. Connection to Bounded Arithmetic Theories

Frege Systems

- Lines are propositional formulas over a complete Boolean basis

- A Frege Rule is a system of formulas

$$\frac{C_1, \ldots, C_k}{D} \qquad \text{where } C_1, \ldots, C_n \models D$$

logically implies

(means can derive D from $C_1, \ldots, C_k$
for any substitution instance of $C_1, \ldots, C_k, D$)

- An axiom is a rule with $k = 0$

- A Frege system $\mathcal{P}$ is implicationally complete
if $\underbrace{A_1, \ldots, A_n \vdash^{\mathcal{P}} B}_{\text{There is a } \mathcal{P}\text{-proof of } B \text{ from } A_1, \ldots, A_n}$ whenever $A_1, \ldots, A_n \models B$

Frege Systems

Frege Systems : Finite number of Axioms and rules; Lines consist of Boolean formulas

2 types:

Hilbert style : each line in proof is a tautology

gentzen style ; Lines are conditional : If A then B

Notation:

$\vdash^{P} B$ There is a P-proof of B

$A_1, \ldots, A_n \vdash^{P} B$ There is a P-derivation of B from $A_1, \ldots, A_n$

$\models B$ B is a tautology

$A_1, \ldots, A_n \models B$ $A_1 \wedge \ldots \wedge A_n$ logically imply B

Example of Hilbert Style : Shoenfield's System

Lines are Boolean formulas over $\wedge, \neg$

Axiom: $A \vee \neg A$

Rules:

$$\frac{A}{A \vee B} \qquad \frac{A \vee A}{A} \qquad \frac{A \vee (B \vee C)}{(A \vee B) \vee C} \qquad \frac{A \vee B, \ \neg A \vee C}{B \vee C}$$

Tait calculus

Lines are $\{B_1, \ldots, B_n\}$ ← A set of Boolean formulas over $\wedge, \vee, \neg$

Interpretation: $B_1 \vee B_2 \vee \ldots \vee B_n$

Axiom: $A, \neg A, \Gamma$

Rules:

$$\frac{\Gamma}{\Gamma, A} \quad \text{(weakening)}$$

$$\frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B} \quad \text{(AND intro)} \qquad \frac{\Gamma, A, B}{\Gamma, A \vee B} \quad \text{(OR intro)}$$

$$\frac{\Gamma, A \qquad \Gamma, \neg A}{\Gamma} \quad \text{(cut rule)}$$

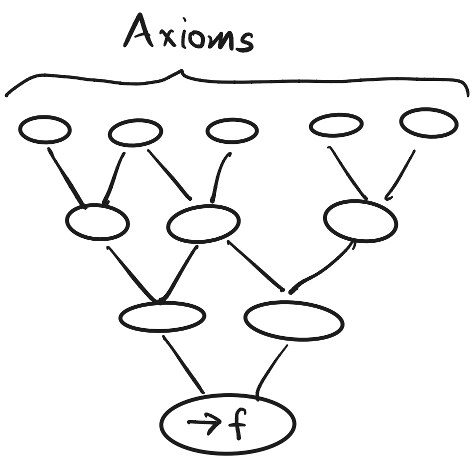
Sequent Calculus

Lines are sequents: $\underbrace{A_1, \ldots, A_n}_{\Gamma} \to \underbrace{B_1, \ldots, B_m}_{\Delta}$

Set of formulas → set of formulas

Meaning: $(A_1 \wedge \cdot \wedge A_n)$ implies $(B_1 \vee \ldots \vee B_m)$

$[A_1 \wedge \ldots \wedge A_n \supset B_1 \vee \ldots \vee B_m]$



Proof dag for
a proof that f is a tautology

Sequent Calculus

Axiom: $A \to A$

Weakening Rule: $\dfrac{\Gamma \to \Delta}{\Gamma, A \to \Delta, B}$

Logical Rules:

AND-RT $\dfrac{\Gamma \to \Delta, A \qquad \Gamma \to \Delta, B}{\Gamma \to \Delta, A \wedge B}$

AND-LEFT $\dfrac{A, B, \Gamma \to \Delta}{A \wedge B, \Gamma \to \Delta}$

OR-RT $\dfrac{\Gamma \to \Delta, A, B}{\Gamma \to \Delta, A \vee B}$

OR-LEFT $\dfrac{A, \Gamma \to \Delta, \qquad B, \Gamma \to \Delta}{A \vee B, \Gamma \to \Delta}$

NEG-RT $\dfrac{\Gamma, A \to \Delta}{\Gamma \to \Delta, \neg A}$

NEG-LEFT $\dfrac{\Gamma \to \Delta, A}{\Gamma, \neg A \to \Delta}$

CUT RULE: $\dfrac{A, \Gamma \to \Delta \qquad \Gamma \to \Delta, A}{\Gamma \to \Delta}$

* Tait Calculus is a simplification of Sequent Calculus where all lines look like $\to B_1, .., B_m$ (empty antecedent)

Soundness / Completeness of Frege

By poly-equivalence of Frege systems, it suffices to prove soundness + completeness for a particular Frege system — we'll do sequent calculus / Tait proofs

Soundness: Follows by induction using implicational soundness of all rules + axioms

Completeness: we will prove that any tautology f has a cut-free sequent calculus proof

Idea: Start with $\rightarrow f$ at root, apply cut-free axioms to some outermost connective until we eventually get $\Gamma, A \rightarrow \Delta, A$ at each leaf.

Completeness of Sequent Calculus – Example

Let $f = \neg(\neg P \wedge \neg Q) \vee \neg(P \vee Q)$

OR RT

$$\frac{\rightarrow A, B}{\rightarrow A \vee B}$$

$\rightarrow \neg(\neg P \wedge \neg Q) \vee \neg(P \vee Q)$

OR-Right

$\rightarrow \neg(\neg P \wedge \neg Q), \neg(P \vee Q)$

¬-left

$\neg P \wedge \neg Q \rightarrow \neg(P \vee Q)$

¬-Right

$(P \vee Q), \neg P \wedge \neg Q \rightarrow$

AND-left

$(P \vee Q), \neg P, \neg Q \rightarrow$

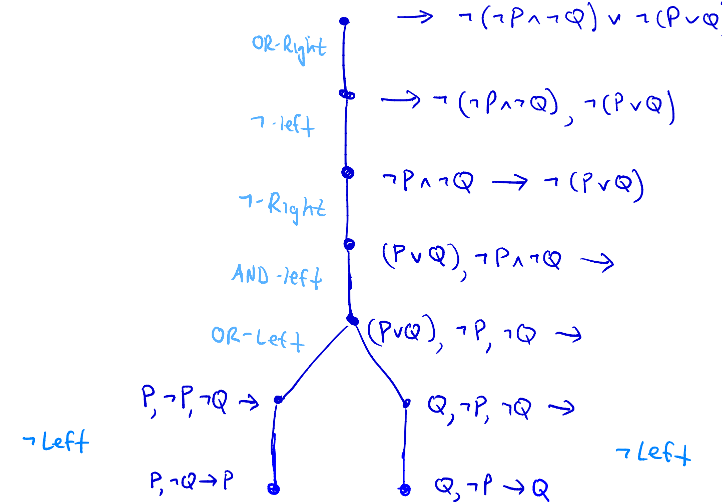
OR-Left

$P, \neg P, \neg Q \rightarrow$ | $Q, \neg P, \neg Q \rightarrow$

¬Left | ¬Left

$P, \neg Q \rightarrow P$ | $Q, \neg P \rightarrow Q$

Completeness of Sequent Calculus – Example



at each step (from root to leaf), the number of connectives in each sequent drops by ≥ 1.

<u>Claim</u> if f is a tautology, along every path in constructed tree, we will eventually reach a leaf where some formula, say A, appears on both the left & right side of sequent.

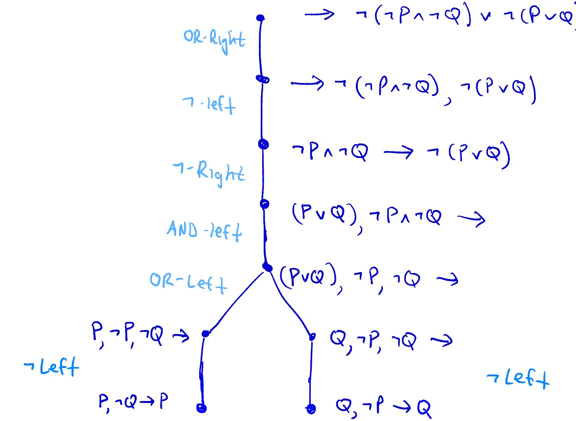
<u>Pf</u> if not, we eventually reach a sequent of the form

$$\underbrace{x_{i_1}, \ldots, x_{i_c}}_{\text{set to } 1} \rightarrow \underbrace{x_{j_1}, \ldots, x_{j_d}}_{\text{set to } 0}$$

where vars on left & right sides are disjoint

But then we can construct a falsifying assignment by setting all x's on Left to 1 & x's on rt to 0.

Completeness of Sequent Calculus – Example



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$$\underbrace{x_{i_1}, \ldots, x_{i_c}}_{\text{set to } 1} \rightarrow \underbrace{x_{j_1}, \ldots, x_{j_d}}_{\text{set to } 0}$$

where vars on left + right sides are disjoint

But then we can construct a falsifying assignment by setting all x's on Left to 1 + x's on rt to 0.

Inversion Property of logical rules:

$$\frac{\Gamma \rightarrow \Delta, A \qquad \Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, A \wedge B}$$

If $\exists$ an assignment α falsifying both upper sequents, then α falsifies lower sequent

(soundness is other direction: if top sequents both satisfiable, so is bottom sequent)

By inversion property this implies that there is an assignment that falsifies f.

$\mathcal{C}$ - Frege

restrict cut formula $A \in \mathcal{C}$

Axiom: $A \to A$

Weakening Rule: $\dfrac{\Gamma \to \Delta}{\Gamma, A \to \Delta, B}$

Logical Rules:

AND-RT $\dfrac{\Gamma \to \Delta, A \qquad \Gamma \to \Delta, B}{\Gamma \to \Delta, A \wedge B}$

AND-LEFT $\dfrac{A, B, \Gamma \to \Delta}{A \wedge B, \Gamma \to \Delta}$

OR-RT $\dfrac{\Gamma \to \Delta, A, B}{\Gamma \to \Delta, A \vee B}$

OR-LEFT $\dfrac{A, \Gamma \to \Delta, \qquad B, \Gamma \to \Delta}{A \vee B, \Gamma \to \Delta}$

NEG-RT $\dfrac{\Gamma, A \to \Delta}{\Gamma \to \Delta, \neg A}$

NEG-LEFT $\dfrac{\Gamma \to \Delta, A}{\Gamma, \neg A \to \Delta}$

CUT RULE: $\dfrac{A, \Gamma \to \Delta \qquad \Gamma \to \Delta, A}{\Gamma \to \Delta}$

Unbounded-fanin Frege (as Sequent Calculus)

all Nonlogical rules, axioms + cut rule + Logical Rules

Logical Rules

$\neg$ Left $\quad \dfrac{\Gamma \to \Delta, A}{\neg A, \Gamma \to \Delta}$ $\qquad$ $\neg$ Right $\quad \dfrac{A, \Gamma \to \Delta}{\Gamma \to \neg A, \Delta}$

AND-Left $\quad \dfrac{A_1, \wedge(A_2, \ldots, A_n), \Gamma \to \Delta}{\wedge(A_1, \ldots, A_n), \Gamma \to \Delta}$ $\qquad$ AND-RT $\quad \dfrac{\Gamma \to A_1, \Delta \qquad \Gamma \to \wedge(A_2 \ldots A_n), \Delta}{\Gamma \to \wedge(A_1, \ldots, A_n), \Delta}$

OR-LEFT $\quad \dfrac{A_1, \Gamma \to \Delta \qquad \wedge(A_2 \ldots A_n), \Gamma \to \Delta}{\vee(A_1, \ldots A_n), \Gamma \to \Delta}$ $\qquad$ OR-RT $\quad \dfrac{\Gamma \to A_1, \vee(A_2, \ldots, A_n), \Delta}{\Gamma \to \vee(A_1, \ldots, A_n), \Delta}$

AC^0-Frege: Unbdded fanin Frege proof, with restriction that all formulas in proof are in AC^0.

$AC^0[2]$ - Frege

Logical Rules

$\neg$ Left $\dfrac{\Gamma \to \Delta, A}{\neg A, \Gamma \to \Delta}$ $\neg$ Right $\dfrac{A, \Gamma \to \Delta}{\Gamma \to \neg A, \Delta}$

AND-Left $\dfrac{A_1, \wedge(A_2, \ldots, A_n), \Gamma \to \Delta}{\wedge(A_1, \ldots, A_n), \Gamma \to \Delta}$ AND-RT $\dfrac{\Gamma \to A_1, \Delta \quad \Gamma \to \wedge(A_2 \ldots A_n), \Delta}{\Gamma \to \wedge(A_1, \ldots, A_n), \Delta}$

OR-LEFT $\dfrac{A_1, \Gamma \to \Delta \quad \wedge(A_2 \ldots A_n), \Gamma \to \Delta}{\vee(A_1, \ldots A_n), \Gamma \to \Delta}$ OR-RT $\dfrac{\Gamma \to A_1, \vee(A_2, \ldots, A_n), \Delta}{\Gamma \to \vee(A_1, \ldots, A_n), \Delta}$

same as AC^0 Frege

$\oplus_2$-left

$$\frac{A_1, \oplus_{b-1}(A_2 \ldots A_n), \Gamma \to \Delta \qquad \oplus_b(A_2 \ldots A_n), \Gamma \to A_1, \Delta}{\oplus_b(A_1, \ldots, A_n), \Gamma \to \Delta}$$

$\oplus_2$-right

$$\frac{A_1, \Gamma \to \oplus_{b-1}(A_2 \ldots A_n), \Delta \qquad \Gamma \to A_1, \oplus_b(A_2 \ldots A_n), \Delta}{\Gamma \to \oplus_b(A_1 \ldots A_n), \Delta}$$

Cook-Reckhow: Robustness of Frege Systems

* Cook-Reckhow (see Reckhow's Thesis) proved that essentially all Frege systems (even over different complete fanin-O(1) bases) are polynomially equivalent

- Proofs involve many manipulations which show how to efficiently translate proofs in one Frege system to another Frege system

Polynomial Equivalences for Frege Systems:

poly equiv

- Natural deduction
- Gentzen systems (sequent calculus, tait calculus)
- Shoenfield
- ⋮

Normal Form for Frege Proofs

Theorem (Frege Normal Form) Let Π be a Frege proof of f. Then there exists another Frege proof Π' of f such that:

(1) Π' is balanced and tree-like

(2) $|\Pi'| \le \text{poly}(|\Pi|)$

Frege Normal Form Proof Sketch

1. Conversion to tree-like:

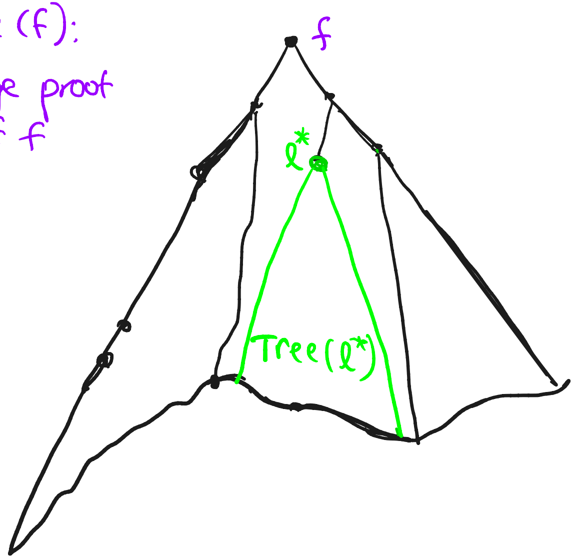
- Replace each line f_i with $f_1 \wedge f_2 \wedge \dots \wedge f_i$
- give a tree-like proof of $\bigwedge_{j=1}^{i+1} f_j$ from $\bigwedge_{j=1}^{i} f_j$

Frege Normal Form Proof Sketch

(2.) conversion from tree-like to balanced, tree-like

Tree(f):

Frege proof of f



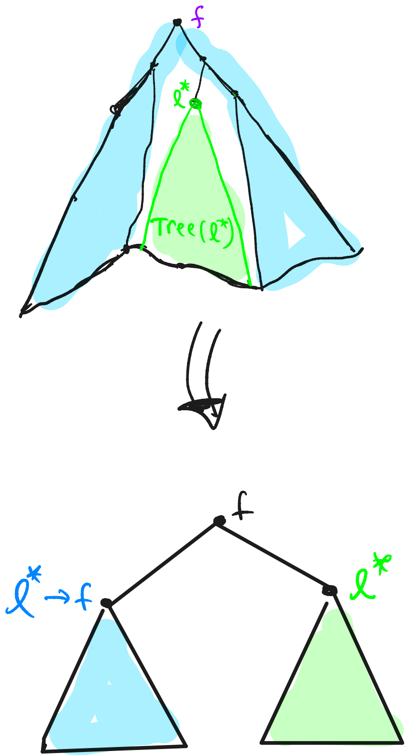
(a) Find line ℓ^* st.

$$\frac{1}{3}|\mathrm{Tree}(f)| \leq |\mathrm{Tree}(\ell^*)| \leq \frac{2}{3}|\mathrm{Tree}(f)|$$

Frege Normal Form Proof Sketch

2. conversion from tree-like to balanced, tree-like

Tree(f): Frege proof of f



(b) Inductively give balanced derivations of

(i) $\ell^* \to f$

(ii) ℓ^*

Then use cut rule on (i) & (ii) to derive f

More fine grained Relationship (tree vs dag like)

Theorem Tree-like AC^0_{d+1} - Frege is poly-equivalent to

AC^0_d - Frege (dag like)

Cook-Reckhow: Robustness of EXTENDED Frege

Later papers proved similar equivalences for Extended Frege Systems

Extended Frege : Frege + Axiom $y \leftrightarrow A(\vec{x})$

where y is a new variable

Allows Lines to be represented more succinctly as Boolean circuits

Polynomial Equivalences for Extended Frege :

poly equiv

- Extended Frege
- Substitution Frege [Dowd]
- Renaming Frege, 0/1 substitution Frege [Buss]
- Hajos calculus [P-Urquhart]

Cook-Reckhow: Robustness of EXTENDED Frege

Polynomial Equivalences for Extended Frege:

poly equiv

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Open: Permutation Frege relative complexity

Frege $\leq$ Permutation Frege $\leq$ Extended Frege

Frege Systems: Equivalent Formulation as Prover - Delayer game

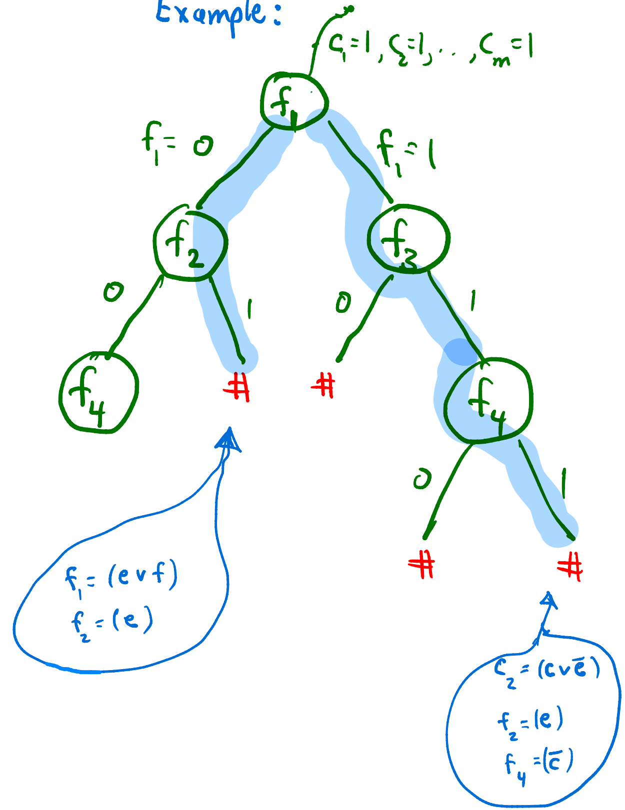
Frege Prover - Liar game:

Liar claims he has a satisfying assignment α for $f = C_1 \wedge C_2 \wedge \ldots \wedge C_m$

Prover queries arbitrary formulas $f_1, \ldots f_q$

game ends when every path has a "truth table" contradiction

Example:

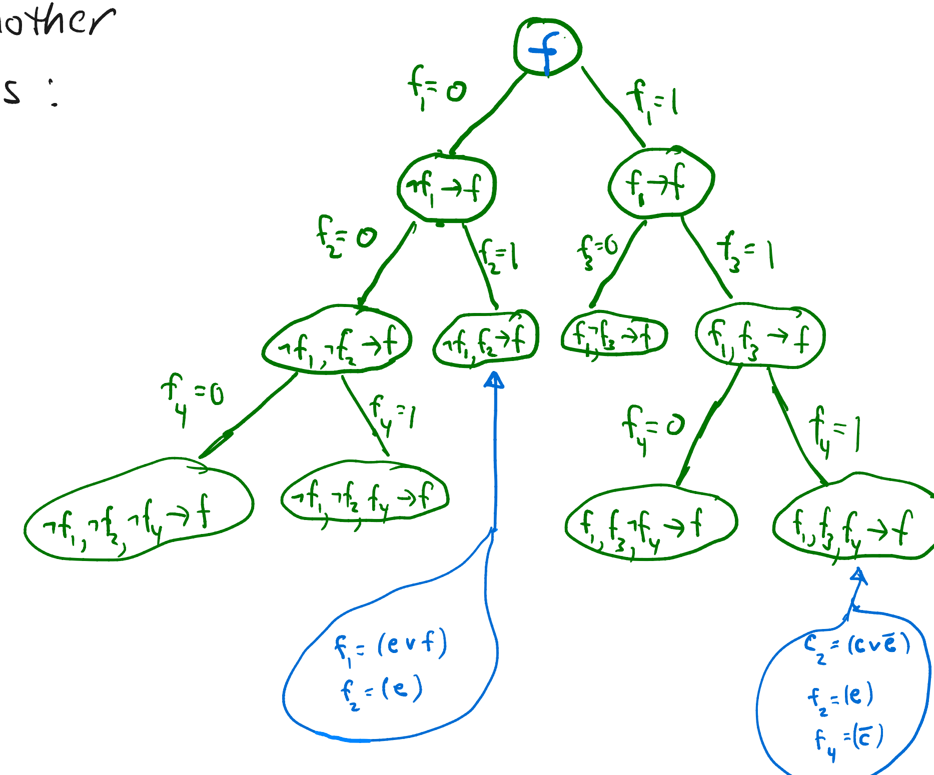


Frege Systems: Equivalent Formulation as Prover-Delayer game

Note this implicitly gives another normal form for Frege proofs:

Proof consists of a tree of cuts, with simple derivations at leaves.

Example:



"Complete" Axiom for Frege / EF

Theorem Let $\text{Soundness}_{\text{Frege}}(s)$ be the propositional formula:

$$\underbrace{\text{Proof}_{\text{Frege}}(\vec{f}, \vec{\pi})}_{\text{states that } \vec{\pi} \text{ is a size } s \text{ proof of } f} \rightarrow \underbrace{\neg \text{SAT}(\vec{f}, \vec{\alpha})}_{\text{states that } \vec{\alpha} \text{ is not a satisfying assignment for } f}$$

Then (1) $\text{Soundness}_{\text{Frege}}(s)$ has a polysized Frege Proof

(2) For any pf system P that poly-simulates depth-2 Frege

$P + \text{Soundness}_{\text{Frege}}(s)$ can poly-simulate Frege

"Complete" Axiom for Frege / EF

Same theorem also holds for EF.

Therefore Frege + Soundness$_{EF}$ $\equiv_{\text{poly}}$ EF

$\therefore$ Frege p-simulates EF iff

Frege has polysized proofs of Soundness$_{EF}$

Avigad ["Plausibly Hard Combinatorial Tautologies"]

shows that there is a natural combinatorial principle

that is poly-equivalent to Soundness$_{EF}$

References

1. Cook-Reckhow and Reckhow thesis

2. Buss paper (Renaming Frege = EF = 0/1 Subs. Frege = Subst Frege)
"Substitution and Propositional Pf complexity" + references therein

3. Pudlak-Buss (Frege as Liar game)
"How to Lie without being (easily) convicted and the lengths of proofs in the propositional calculus"

4. Urquhart-Pitassi : "Complexity of Hajos Calculus" (EF = Hajos Calc)

5. Avigad Paper (combinatorial Principle equiv to EF soundness)
"Plausibly Hard Combinatorial Tautologies"

6. Completeness of sequent calculus : Cook-Pitassi Lecture Notes
Logic + Computability Fall 2022