Propositional Proof Complexity
Assignment # 1
Due: Monday March 24, 2025

1. Give a Sequent Calculus proof of the following sequent.

   $(x_1 \vee x_2), (y_1 \vee y_2), (z_1 \vee z_2) \to (x_1 \wedge y_1), (x_1 \wedge z_1), (y_1 \wedge z_1), (x_2 \wedge y_2), (x_2 \wedge z_2), (y_2 \wedge z_2)$

   Hint: you may use the procedure described in class for proving the completeness of the Sequent Calculus.

2. Prove that any unsatisfiable 2CNF formula has a Resolution refutation of polynomial size.

3. The mod 2 counting principle, $\mathrm{Mod2}_n$ asserts that there is no perfect matching on an odd number of vertices. The negation of the mod 2 counting principle, $\neg\mathrm{Mod2}_n$ is a CNF formula with underlying variables $x_{i,j}$ for $i \neq j$, $i, j \leq 2n+1$ to represent whether or not there is a matching between vertices i and j. The clauses of $\neg\mathrm{Mod2}_n$ are of two types:

   (i) For every $i \leq 2n+1$ we have the clause $(\vee_{j \neq i} x_{i,j})$ stating that each vertex is included in at least one matching.

   (ii) Secondly, for every $i, j, k \leq 2n+1$, $i \neq j \neq k$, we have the clause $(\neg x_{ij} \vee \neg x_{i,k})$, stating that every vertex i is matched with at most one other vertex.

   Prove that for n sufficiently large, any tree-like Resolution refutation of $\neg\mathrm{Mod2}_n$ requires size $2^{\Omega(n)}$.

   Hint: Consider the proof that we did in class showing that tree-like Resolution refutations of the pigeonhole principle require size $2^{\Omega(n)}$, and try to use a similar argument here.

4. Let F be an unsatisfiable 3CNF over variables $z_1, \ldots, z_n$. The formula $F \circ \oplus^n$ is a new 6CNF formula a on variables $x_1, y_1, \ldots, x_n, y_n$ as follows. First substitute each variable z_i in F by the expression $x_i \oplus y_i$ where $\oplus$ is the XOR function. Then re-write the substituted formula as a 6CNF formula. Note that if F has width w and s clauses, then $F \circ \oplus^n$ will have with $2w$ and at most $s2^w$ clauses. Prove any Resolution refutation of $F \circ \oplus^n$ has size $2^{\Omega(\mathbf{w}(F))}$, where $\mathbf{w}(F)$ is the minimal width of any Resolution refutation of F.

   Hint: Consider the family of random restrictions R^n where a random $\rho \in R^n$ is selected as follows: for each $i \leq n$, select one of x_i or y_i with equal probability, and then set the selected variable to either 0 or 1 with equal probability. Prove using the probabilistic method that if $F \circ \oplus^n$ has a Resolution refutation Π of size $s < 2^{\Omega(\mathbf{w}(F))}$, then there exists a random restriction $\rho \in R^n$ such that $\Pi|_\rho$ has width less than $\mathbf{w}$.

5. (Extra Credit) Improve your lower bound for Question 3 above by showing that any tree-like Resolution refutation of $\neg\mathrm{Mod}2_n$ requires size $2^{\Omega(n \log n)}$.