Lecture 7

- HW1 due (Tues) on gradescope

- Today: DFA minimization, practice problems, wrap up Regular languages

Regular Languages – Summary

1. DFAs and Regular Languages

2. NFAs, and equivalence with DFAs

3. Closure Properties of Regular Languages

4. Regular Expressions and Equivalence with Regular Languages

5. Proving that a language is NOT regular:
   Pumping Lemma

~~6.~~ DFA state minimization ← we won't cover

Other Practice Problems

1. Construct a DFA M accepting the language corresponding to the regular expression $(0+1)^*011(0+1)^*$

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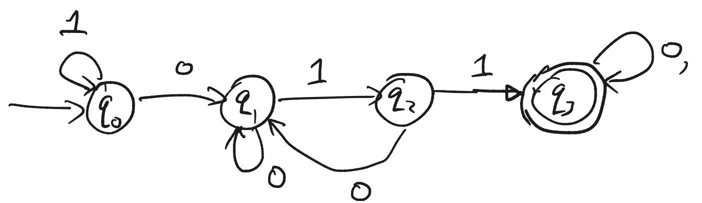
DFA:

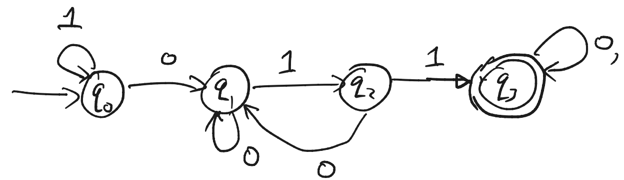


q_0: Not began to see the substring 011
q_1: we have seen the beginning 0
q_2 " " " " " 1
q_3 " " " the entire substring 011

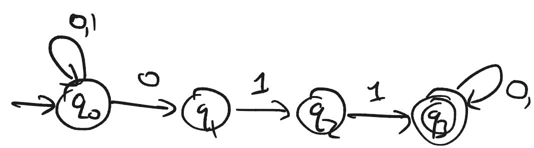
Other Practice Problems

1. Construct a DFA M accepting the language corresponding to the regular expression $(0+1)^* 011 (0+1)^*$

DFA:



NFA:



Other Practice Problems

2. Write regular expressions for the following languages

(a) The set of all $w \in \{0,1\}^*$ that do not contain the substring 00

Other Practice Problems

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$$(1+01)^* \cdot (\varepsilon + 0) = (1+01)^* + (1+01)^* 0$$

$$(0+1)^{*} = (0+1)^{i} \quad \forall i = 0, 1, 2, \ldots, n$$

$(0+1)^2$: $00, 01, 10, 11$

Other Practice Problems

2. Write regular expressions for the following languages

(a) The set of all $w \in \{0,1\}^*$ that do not contain the substring 00

$$(1 + 01)^* (\varepsilon + 0)$$

(b) The set of all $w \in \{0,1\}^*$ that contain at least 3 1's

Other Practice Problems

2. Write regular expressions for the following languages

(a) The set of all $w \in \{0,1\}^*$ that do not contain the substring 00

$$(1+01)^*(\varepsilon+0)$$

(b) The set of all $w \in \{0,1\}^*$ that contain at least 3 1's

$$(0+1)^* 1 (0+1)^* 1 (0+1)^* 1 (0+1)^*$$

(c) w contains exactly 3 1's

$$0^* 1 0^* 1 0^* 1 0^*$$

Other Practice Problems

3. Let $L = \{ a^n \mid n \text{ is prime} \}$

Prove that L is Not regular.

Ex. $w = aaaaa \quad (n=5) \quad \in L$

$w = aaaaaaa \quad (n=7) \quad \in L$

$w = aaaa \quad (n=4) \quad \notin L$

Other Practice Problems

3. Let $L = \{ a^n \mid n \text{ is prime} \}$

Prove that L is Not regular.

Proof Assume for contradiction that L is regular. Let M be a k-state DFA that accepts L

- Let $w = a^p$ where p is a prime number, $p \geq k$.
  Note that $w \in L$.

Other Practice Problems

3. Let $L = \{ a^n \mid n \text{ is prime} \}$

Prove that L is Not regular.

Proof Assume for contradiction that L is regular. Let M be a k-state DFA that accepts L

- Let $w = a^p$ where p is a prime number, $p \geq k$.
  Note that $w \in L$, so w should be accepted by M.
- By Pumping Lemma, we can write $w = xyz$, $|xy| \leq k$, $|y| > 0$ such that for all i, $w' = xy^iz$ is also accepted by M
- Since $|xy| \leq k$ and $p \geq k$, $w = \underbrace{a^u}_{x} \underbrace{a^v}_{y} \underbrace{a^{p-u-v}}_{z}$, $u+v \leq k \leq p$
- Pick $i = p+1$.

Other Practice Problems

3. Let $L = \{ a^n \mid n \text{ is prime} \}$

- Let $w = 1^p$ where p is prime, $p \geq k$. Note that $w \in L$.
- By Pumping Lemma, we can write $w = xyz$, $|xy| \leq k$, $|y| > 0$ such that for all i, $w' = xy^iz$ is also in L.
- Since $|xy| \leq k$ and $p \geq k$, $w = \underbrace{a^u}_{x}\underbrace{a^v}_{y}\underbrace{a^{p-u-v}}_{z}$, $u+v \leq k \leq p$
- Pick $i = p+1$. Then $w' = xy^iz$ $= a^{u+v(p+1)+p-u-v}$

$$= a^{v(p+1)+p-v}$$
$$= a^{vp+p}$$
$$= a^{p(v+1)}$$

$$w' = xy^iz$$
$$= a^{u+vi+p-u-v}$$
$$= vi+p-v$$
$$\underbrace{p+\underbrace{v(i-1)}_{vp}}_{p(v+1)}$$

$\therefore$ $w' \notin L$, so we've reached a contradiction.

Other Practice Problems

4. Let $\Sigma_3 = \left\{ \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \right\}$

$w \in \Sigma_3^*$ looks like

e.g. $\begin{pmatrix}0\\0\\1\end{pmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}$

Let $B = \{ w \in \Sigma_3^* \mid$ the bottom row of w (as a binary number) equals the sum of the top two rows $\}$

Examples: $\begin{pmatrix}0\\0\\1\end{pmatrix} \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} \cdot \begin{pmatrix}1\\1\\0\end{pmatrix} \in B$

$$\begin{array}{rccc} & {}^{1}0 & {}^{1}1 & 1 \\ + & 0 & 0 & 1 \\ \hline & 1 & 0 & 0 \end{array}$$

$\begin{pmatrix}0\\0\\1\end{pmatrix} \cdot \begin{pmatrix}1\\0\\1\end{pmatrix} \notin B$

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Other Practice Problems

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$$\begin{array}{r} 0\,1\,1 \\ +\ 0\,0\,1 \\ \hline 1\,0\,0 \end{array}$$

$\begin{pmatrix}0\\0\\1\end{pmatrix} \cdot \begin{pmatrix}1\\0\\1\end{pmatrix} \notin B$

$$\begin{array}{r} 0\,1 \\ +\ 0\,0 \\ \hline 0\,1 \end{array}$$

Prove that B is regular.

Other Practice Problems

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$$\begin{array}{r} 0\,1\,1 \\ +\;0\,0\,1 \\ \hline 1\,0\,0 \end{array}$$

$\begin{pmatrix}0\\0\\1\end{pmatrix} \cdot \begin{pmatrix}1\\0\\1\end{pmatrix} \notin B$

$$\begin{array}{r} 0\,1 \\ +\;0\,0 \\ \hline 0\,1 \end{array}$$

Idea: show that B^R is regular (and therefore so is B)

In general if L is a language $L \subseteq \Sigma^*$,

$$L^R = \{ w \mid w^R \in L \}$$

Example

Let $L = \{ w \mid w \text{ starts with a } 1 \}$ (over $\{0,1\}$)

$L^R = \{ w \mid w \text{ ends in a } 1 \}$

$$\left.\begin{array}{ll} w_1^R = 0001 \in L^R & w_1 = 1000 \\ w_2^R = 0011 \in L^R & w_2 = 1100 \end{array}\right\} \in L$$

$w_3^R = 11110 \notin L^R \qquad w_3 = 01111 \notin L$

Other Practice Problems

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Let $B = \{ w \in \Sigma_3^* \mid$ the bottom row of w (as a binary number) equals the sum of the top two rows $\}$

Examples: $w = \begin{pmatrix}0\\0\\1\end{pmatrix} \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} \cdot \begin{pmatrix}1\\1\\0\end{pmatrix} \in B$

$$w^R = \begin{matrix}1 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{matrix} \qquad \begin{array}{r} 1\;1 \\ 1\;1\;0 \\ 1\;0\;0 \\ \hline 0\;0\;1 \end{array}$$

$w = \begin{pmatrix}0\\0\\1\end{pmatrix} \cdot \begin{pmatrix}1\\0\\1\end{pmatrix} \notin B$

$$w^R = \begin{matrix}1 & 0\\ 0 & 0\\ 1 & 1\end{matrix} \qquad \begin{array}{r} 1\;0 \\ 0\;0 \\ \hline 1\;1 \end{array}$$

Idea: show that B^R is regular (and therefore so is B)

① Show: $\forall$ Language $L \subseteq \Sigma^*$,

L is regular if and only if L^R is regular.

Other Practice Problems

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Examples: $w = \begin{pmatrix}0\\0\\1\end{pmatrix} \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} \cdot \begin{pmatrix}1\\1\\0\end{pmatrix} \in B$ $\quad w^R = \begin{matrix}1 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{matrix}$ $\quad \begin{array}{ccc} 1 & 1 & \\ 1 & 1 & 0\\ 1 & 0 & 0\\ \hline 0 & 0 & 1\end{array}$

$w = \begin{pmatrix}0\\0\\1\end{pmatrix} \cdot \begin{pmatrix}1\\0\\1\end{pmatrix} \notin B$ $\quad w^R = \begin{matrix}1 & 0\\ 0 & 0\\ 1 & 1\end{matrix}$ $\quad \begin{array}{cc} 1 & 0\\ 0 & 0\\ \hline 1 & 1\end{array}$

Idea: show that B^R is regular (and therefore so is B)

Other Practice Problems

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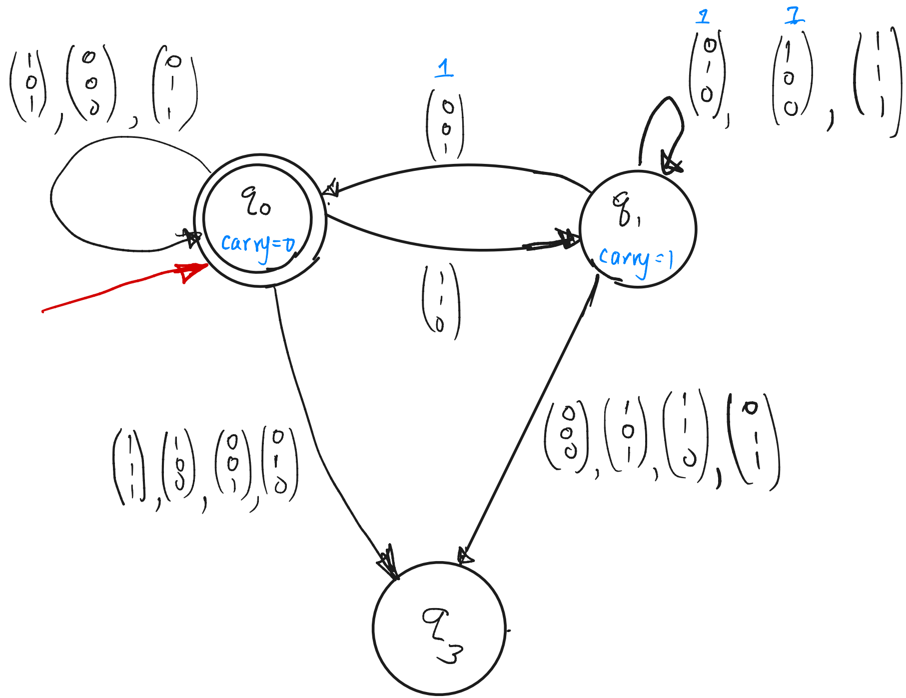

← accepts B^R

Other Practice Problems

5. State if each of the inequalities is True / False. Justify your answer.

(a) $(rs+r)^* = r(sr+r)^*$

$$(0+1)^{\geq 1} = (0+1)(0+1)^*$$

Ex 1

$$rs \in (rs+r)^*$$

$$rs \notin r(sr+r)^*$$

Ex 2

$$\epsilon \in (rs+r)^*$$

$$\epsilon \notin r(sr+r)^*$$

Other Practice Problems

5. State if each of the inequalities is True / False. Justify your answer.

(a) $(rs+r)^* = r(sr+r)^*$

Not equal.

$\varepsilon \in (rs+r)^*$, $\varepsilon \notin r(sr+r)^*$

$rs \in (rs+r)^*$, $rs \notin r(sr+r)^*$

Other Practice Problems

5. State if each of the inequalities is True/False. Justify your answer.

(b) $s \cdot (rs + s)^* = r^* \cdot s\,(r \cdot r^* \cdot s)^*$

$rs \notin s(rs+s)^*$

$rs \in$

Other Practice Problems

5. State if each of the inequalities is True/False. Justify your answer.

(b) $s\cdot(rs+s)^* = r^* s\,(r\cdot r^*\cdot s)^*$

No $rs \notin s(rs+s)^*$

$rs \in rs(rr^*s)^*$

$srrs \in r^*s\,(rr^*s)^*$, $srrs \notin s(rs+s)^*$

Other Practice Problems

5. State if each of the inequalities is True / False. Justify your answer.

(c) $\underbrace{(r+s)^*}_{}$ $= (r^* + s^*)^*$

all possible $w \in \{r, s\}^*$

$L[(r'+s')^*]$

$\subseteq L[(r^*+s^*)^*]$

so the language on rt $L[(r^*+s^*)^*]$, contains the language on the left side, $L[(r+s)^*]$

On the other hand, $L[(r+s)^*]$ is all strings, so same.

Other Practice Problems

5. State if each of the inequalities is True / False. Justify your answer.

(c) $(r+s)^* = (r^* + s^*)^*$

Yes. Both Languages are the same:

(1) $\mathcal{L}[(r+s)^*] \subseteq \mathcal{L}[(r^*+s^*)^*]$:

$$\mathcal{L}[(r)] \subseteq \mathcal{L}[(r^*)], \quad \mathcal{L}[(s)] \subseteq \mathcal{L}[(s^*)]$$

$$\therefore \mathcal{L}[(r+s)] \subseteq \mathcal{L}[(r^* + s^*)]$$

$$\therefore \mathcal{L}[(r+s)^*] \subseteq \mathcal{L}[(r^* + s^*)^*]$$

Other Practice Problems

5. State if each of the inequalities is True/False. Justify your answer.

(c) $(r+s)^* = (r^* + s^*)^*$

(ii) $\mathcal{L}[(r^* + s^*)^*] \subseteq \mathcal{L}[(r+s)^*]$:

Since $\mathcal{L}[(r+s)^*] = \{r, s\}^*$ = all strings over $\Sigma = \{a, b\}$,

it follows that $\mathcal{L}[(r^* + s^*)^*] \subseteq \mathcal{L}[(r+s)^*]$

(since every language is a subset of $\{r, s\}^*$)