Lecture 6

- HW1 due next Tues.
- Homework tips
- Pumping lemma

1. #3 Labelled Hard.

Assume A is Regular language.
Show that Half(A) is also regular.

$$Half(A) = \{ x \mid \exists y \text{ s.t. } |x| = |y| \text{ and } xy \in A \}$$

(Assume L_1, L_2 are regular.
Prove $L_1 + L_2$
$L_1 \cdot L_2$
are regular)

2. accept all strings w such that every 4 consecutive symbols of w contains at least 2 0's.

NFA (suggest DFA)

If you give a DFA, describe transition function by <u>both</u> diagram and English description

Tip / Pitfall

If your construction uses closure properties

then be sure you dont take

an NFA for L' &

use conversion (Δ accept states → reject states
reject " → accept)

from class to go from $L' \longrightarrow \overline{L'}$

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condition

0111
1011
1110
1101

} avoid these as substrings

111 ← accepted

1011 rejected

000111 rejected

<u>Recap so far</u>

1. DFAs and Regular Languages

2. NFAs, and equivalence with DFAs

3. Closure Properties of Regular Languages

4. Regular Expressions and Equivalence with Regular Languages

Next:

5. Proving that a language is <u>Not</u> regular:
   Pumping Lemma

6. DFA state minimization

Nonregular Languages & Pumping Lemma

Warmup: Which of these Languages is regular?

$A = \{ 0^n 1^n \mid n \geq 0 \}$

$B = \{ w \in \{0,1\}^* \mid$ w has equal number of 0's & 1's $\}$

$C = \{ w \in \{0,1\}^* \mid$ w has equal number of occurrences of '01' and '10' $\}$

Nonregular Languages & Pumping Lemma

Warmup: Which of these languages is regular?

$L_1 = \{w \in \{0,1\}^* \mid$ the number of 0's in w is equal to the number of 1's in $w\}$

$L_2 = \{w \in \{0,1\}^* \mid$ the number of occurrences of '01' in w is equal to the number of occurrences of '10' $\}$

Lower Bounds: How to prove that a language is not regular?

$L = \{0^n 1^n \mid n \geq 0\}$

Tricky since we need to show that every DFA M has to make a mistake with respect to L

(Show: either $\exists w \in L$ not accepted by M, or $\exists w$ accepted by M and not in L)

And there are an infinite number of DFAs!

Lower Bounds: How to prove that a language is not regular?

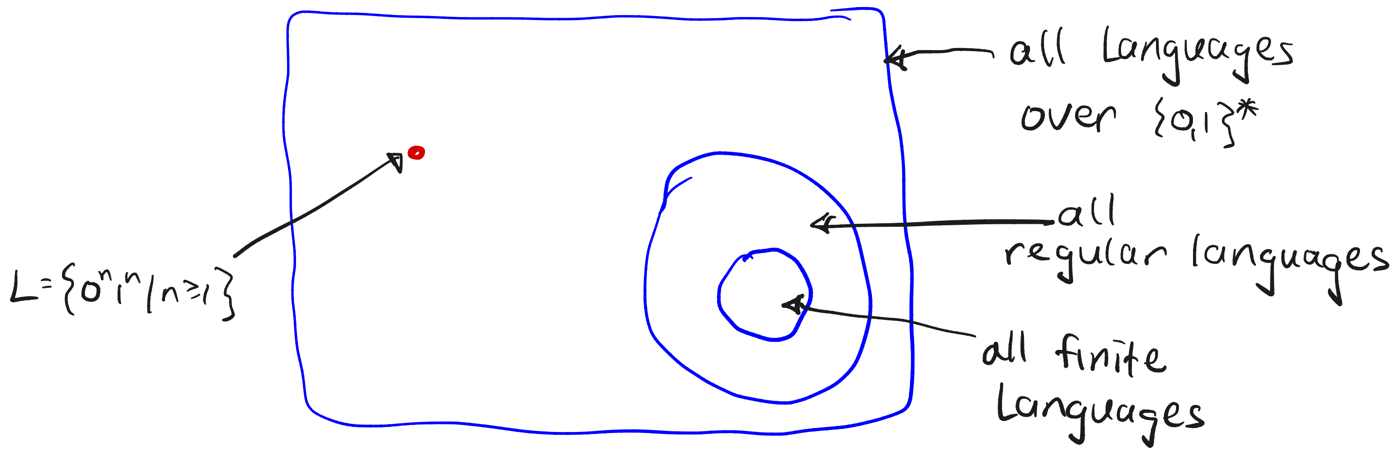
$L = \{0^n 1^n \mid n \geq 0\}$

- Not enough to show that the obvious or natural DFAs don't accept L
- Avoid a common trap:

  L may be defined by some property.
  But we can't assume that a DFA for L needs to be able to recognize/compute that property

Example: $L_2 = \{w \in \{0,1\}^* \mid$ the number of occurrences of '01' in w is equal to the number of occurrences of '10' $\}$

Lower Bounds: How to prove that a Language is not regular?



Proof by contradiction: Assume that L is regular, so some DFA, M, accepts A.

Find some property that all regular Languages have that L doesn't have to get a contradiction.

WARMUP : A Language L' is finite if $\exists c \geq 0$ such that $|L| \leq c$

Let's show : $L = \{0^n 1^n \mid n \geq 0\}$ is not a finite Language.

Property : A language L' is length-bounded if $\exists k \geq 0$ such that every $w \in L'$ has length $\leq k$

Claim All finite languages are length-bounded.

Proof that $L = \{0^n 1^n \mid n \geq 0\}$ is not finite :

- Assume for sake of contradiction that L is finite
- By claim, $\exists k \geq 0$ such that every $w \in L$ has length $\leq k$.
- But $w = 0^k 1^k \in L$ and $|w| > k$. $\therefore L' \neq L$. So L is not finite

Now we will show that $L = \{0^n 1^n \mid n \geq 0\}$ is Not regular

Main tool: Pumping Lemma.

Key Idea Every DFA has a finite number of states.
Therefore for any DFA M (allegedly accepting language L) for any sufficiently large $w \in \Sigma^*$, M's computation on w will Loop.

For example, suppose M has K states. Then for every $w \in \Sigma^*$ of length $\geq K$ ($|w| \geq K$), M will loop on w.

Key Idea Every DFA has a finite number of states.
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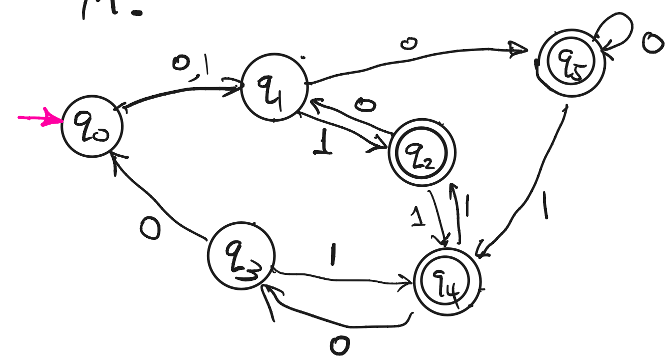
Example

M has $K = 6$ states

So any string w of length ≥ 6 will Loop (repeat a state)

M:



$w = 1011011$ $\quad q_0\ q_1\ q_5\ q_4\ q_2\ q_1\ q_2\ q_4$

$w = 111001$ $\quad q_0\ q_1\ q_2\ q_4\ q_3\ q_0\ q_1$

$w = 100111$ $\quad q_0\ q_1\ q_5\ q_4\ q_2\ q_4\ q_2$

Proof that $L = \{0^n 1^n \mid n \geq 0\}$ is not regular

Property: For any DFA M, $\exists K \geq 0$ such that for every $w \in \Sigma^*$, $|w| \geq K$, M on w repeats a state.

That is, $\forall w$, $|w| \geq K$, $\exists$ a state q^* satisfying:

We can write $w = xyz$, $|y| > 0$, $|xy| \leq K$ satisfying:

M is in state q^* after reading x, and again is in state q^* after reading xy

Therefore for every $i \geq 0$, the string $w' = xy^i z$ behaves the same as w on M. That is:

M accepts w' if and only if M accepts w

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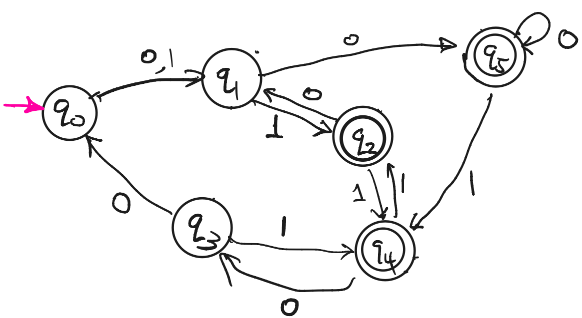
Example

M: $K = 6$



$w = 1011011$ $\quad \underbrace{q_0}_{x} \underbrace{q_1\, q_5\, q_4\, q_2\, q_1}_{y} \underbrace{q_2\, q_4}_{z}$

$w = 111001$ $\quad \underbrace{\varepsilon}_{x} \underbrace{q_0\, q_1\, q_2\, q_4\, q_3\, q_0}_{y} \underbrace{q_1}_{z}$

$w = 1001111$ $\quad \underbrace{q_0\, q_1\, q_5}_{x} \underbrace{q_4\, q_2\, q_4}_{y} \underbrace{q_2}_{z}$

Property: For any DFA M, $\exists p \geq 0$ such that for every $w \in \Sigma^*$, $|w| \geq p$, M on w repeats a state. That is, $\forall w$, $|w| \geq p$, $\exists$ a state q^* satisfying:

We can write $w = xyz$, $|y| > 0$, $|xy| \leq p$ such that M is in state q^* after reading x, and again is in state q^* after reading xy

Therefore for every $i \geq 0$, the string $w' = xy^i z$ behaves the same as w on M. That is: M accepts w' if and only if M accepts w

Proof that $L = \{0^n 1^n \mid n \geq 0\}$ is not regular:

Assume that L is regular & Let M be a DFA accepting L, where M has p states

Consider the input $w = 0^p 1^p$. Since $w \in L$, M should accept w (otherwise we reach contradiction)

By above property, we can write $w = xyz$, $|y| > 0$, $|xy| \leq p$ such that $\forall i \geq 0$ $xy^i z$ is also accepted by M (since M accepts w)

Since $|xy| \leq p$, $|y| > 0$, $w = xyz = \underbrace{0^a}_{x} \underbrace{0^b}_{y} \underbrace{0^{p-a-b} 1^p}_{z}$ $\quad b > 0$

$\therefore$ If w accepted by M, then the string $xy^2z = 0^a 0^{2b} 0^{p-a-b} 1^p = 0^{p+b} 1^p$ is also accepted by M

But $xy^2z \notin L$. Contradiction. $\therefore L$ is not regular.

Note: our proof that xy^2z was fairly simple. This is because we chose w wisely.

$w = 0^p 1^p$ $\quad w = xyz$ $\quad$ * xy contains only 0's since $|xy| \le p$

→ Say we picked instead: $w = 0^{p/2} 1^{p/2}$

Pumping Lemma says: $\quad w = xyz \quad |xy| \le p, \quad |y| \ge 1$

Now there are 3 different cases of how w can be divided into x, y, z satisfying $|xy| \le p$, $|y| \ge 1$

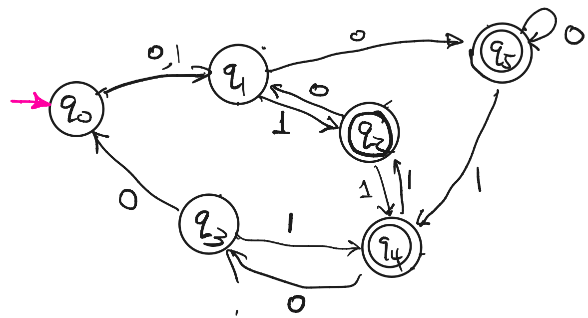
Case 1: $\underbrace{0^a}_{x} \underbrace{0^b}_{y} \underbrace{0^{\frac{p}{2}-a-b} 1^{p/2}}_{z}$

Case 2 $\underbrace{0^a}_{x} \underbrace{0^{\frac{p}{2}-a} 1^c}_{y} \underbrace{1^{\frac{p}{2}-c}}_{z}$

Case 3: $\underbrace{0^{p/2} 1^a}_{x} \underbrace{1^b}_{y} \underbrace{1^{\frac{p}{2}-a-b}}_{z}$

Proof that $L = \{0^n 1^n \mid n \geq 0\}$ is not regular, cont'd

For example:

M: 

$p = 6$

our string $w = 0^p 1^p = 0^6 1^6 = 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1$

M on w accepts: $\underbrace{q_0\ q_1}_{x}\ \underbrace{q_5}_{y}\ \underbrace{q_5\ q_5\ q_5\ q_5\ q_4\ q_2\ q_4\ q_2\ q_4\ q_2}_{z}$

M on $\underbrace{xy^2z}_{\text{"pumped" string}}$: also accepts, but $xy^2z \notin L$

Another example

Let $L' = \{w \in \{0,1\}^* \mid w$ contains the same number of 0's as 1's$\}$

Proof 1 (using Pumping Lemma)

Assume for sake of contradiction that L' is regular, and let M be a DFA accepting L' with $p \geq 0$ states.

Consider $w = 0^p 1^p$. w is in L'.

By Pumping Lemma (same as previous example), w can be written as

$w = x\,y\,z = \underbrace{0^a}_{x}\underbrace{0^b}_{y}\underbrace{0^{p-a-b}\;1^p}_{z}$ for some $b > 0$, and $w' = xy^2z$ is also accepted

by M. But $w' \notin L$, & therefore we reach a contradiction

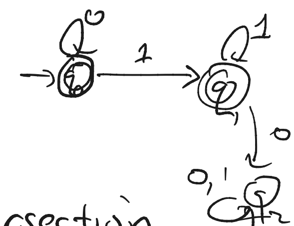
Another example

Let $L' = \{w \in \{0,1\}^* \mid w \text{ contains the same number of 0's as 1's}\}$

Proof 2 (via a reduction, using closure property)

Let $L = \{0^n 1^n \mid n \geq 0\}$.

- we already showed that L is not regular using pumping lemma.
- Also, we have: $L = L' \cap L''$ where $L'' = 0^* 1^*$



Claim (i): L'' is regular

Claim (ii): Regular Languages are closed under intersection.

So if L' is regular, then since L'' is also regular this would imply L regular, but we already proved that L is not regular

$\therefore$ L' is not regular

Here lets try a different choice of w that doesn't work
(to prove that L' is not regular):

$L' = \{w \mid w \text{ has the same number of 0's and 1's}\}$

A bad choice of w: $w = 0^{p/2} 1^{p/2}$

$w = x\ y\ z$

Case 1 $\quad \underbrace{0^a}_{x}\ \underbrace{0^b}_{y}\ \underbrace{0^{p/2-a-b}\ 1^{p/2}}$

Case 2 $\quad \underbrace{0^a}_{x}\ \underbrace{0^{p/2-a}\ 1^b}_{y}\ \underbrace{1^{p/2-b}}_{z}$

Case 3 $\quad \underbrace{0^{p/2}\ 1^a}_{x}\ \underbrace{1^b}_{y}\ \underbrace{1^{p/2-a-b}}_{z}$

in this case y could have the same number of 0's as 1's so no pumping of w in this case will always give us a string w' that is not in L'

Let $L_1 = \{0^i 1^j \mid i < j\}$. L is Not regular.

Pf Assume for sake of contradiction that L_1 is regular, and let M be a DFA with p states that accepts L_1.

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Pf Assume for sake of contradiction that L_1 is regular, and let M be a DFA with p states that accepts L_1.

Let $w = 0^p 1^{p+1}$ Note that $w \in L_1$

By pumping Lemma we can write $w = x\,y\,z$, $|xy| \leq p$, $|y| \geq 1$

such that $w' = x y^2 z$ is also accepted.

Since $|xy| \leq p$ this means $w = \underbrace{0^a}_{x} \underbrace{0^b}_{y} \underbrace{0^{p-a-b} 1^{p+1}}_{z}$ for some $b > 0$

then $w' = x y^2 z = 0^{p+b} 1^{p+1}$

But since $b > 0$, $p + b \geq p+1$ and therefore $w \notin L_1$

Let $L_2 = \{ 0^i 1^j \mid i > j \}$. L is not regular (Example of pumping down)

Pf Assume for sake of contradiction that L_2 is regular, and let M be a DFA with p states that accepts L_2.

Let $w = 0^{p+1} 1^p$. Note that $w \in L$.

By pumping Lemma, we can write $w = x\,y\,z$, $|xy| \leq p$, $|y| \geq 1$

such that $w' = x y^0 z = xz$ is also accepted.

Since $|xy| \leq p$ this means $w = \underbrace{0^a}_{x} \underbrace{0^b}_{y} \underbrace{0^{p+1-a-b} 1^p}_{z}$ $b > 0$

then $w' = x y^0 z = 0^{p+1-b} 1^p$

But since $b > 0$ $p+1-b \leq p$ and therefore $w' \notin L_2$

In summary to prove some language L is not regular (using the Pumping Lemma).

1. Assume for sake of contradiction L is regular.
   (So we are given any M with p states, $p \geq 0$)

2. Based on L, and p
   we choose some w st ① $|w| \geq p$
   ② (typically) $w \in L$

3. Now w is divided into 3 pieces $w = xyz$
   such that $|xy| \leq p$, $|y| > 0$

4. Show we can always pump up w to get a $w' = xy^iz$
   (we pick i) and we need to show $w' \notin L$ (typical case)