Lecture 22

Today : NP, NP-completeness cont'd

Clarifications / Hints on HW3

(2.) Factor : Input: is a number x in decimal

Output: Prime factorization of x

Example: $x = 60$

output : 2, 2, 3, 5

Show: $P = NP \Rightarrow$ FACTOR has a polytime algorithm (in $|x|$)

Clarifications / Hints on HW3

(2.) Factor : Input: is a number x in decimal

Output: Prime factorization of x

Show: $P = NP \Rightarrow$ FACTOR has a polytime algorithm (in $|x|$)

HINT $\boxed{\text{Step 1}}$: FINDFACTOR : Input is x

Output is $\begin{cases} x & \text{if } x \text{ is prime} \\ y_1, y_2 \text{ st } & y_1, y_2 \neq 1 \text{ and } \\ & y_1 y_2 = x \text{ otherwise} \end{cases}$

Show: If FINDFACTOR in polytime then FACTOR in polytime

Clarifications / Hints on HW3

(2.) Factor : Input: is a number x in decimal

Output: Prime factorization of x

Show: $P = NP \Rightarrow$ FACTOR has a polytime algorithm (in $|x|$)

HINT $\boxed{\text{Step 2}}$: show $P = NP \Rightarrow$ FINDFACTOR in poly time

Find a Language L in NP such that

if L is in P then we can solve FINDFACTOR

in polytime (by "binary search")

Say we want ~~to~~ a polynomial time algorithm to solve <u>FIND-$\frac{n}{2}$-CLIQUE</u>

FIND-K-CLIQUE : Input undirected graph g on vertices $x_1 \dots x_n$

Output : "No clique of size $\frac{n}{2}$" if g doesn't contain a $\frac{n}{2}$-clique

~~O~~ ow output a subset $V' \subseteq V$ s.t. V' is a clique in g

show: If $P = NP \implies$ FIND-$\frac{n}{2}$-CLIQUE is in P

Let $A = \left\{ (g, k, V' \subseteq V) \;\middle|\; \right.$ g contains a k-clique including the vertices in V' $\left.\right\}$

g : find a $\frac{n}{2}$-clique:

1. call $A(g, k, V' = \{\cancel{x}_1\})$

Clarifications / Hints on HW3

(3b) This is related to question 1. (P is closed under complement)

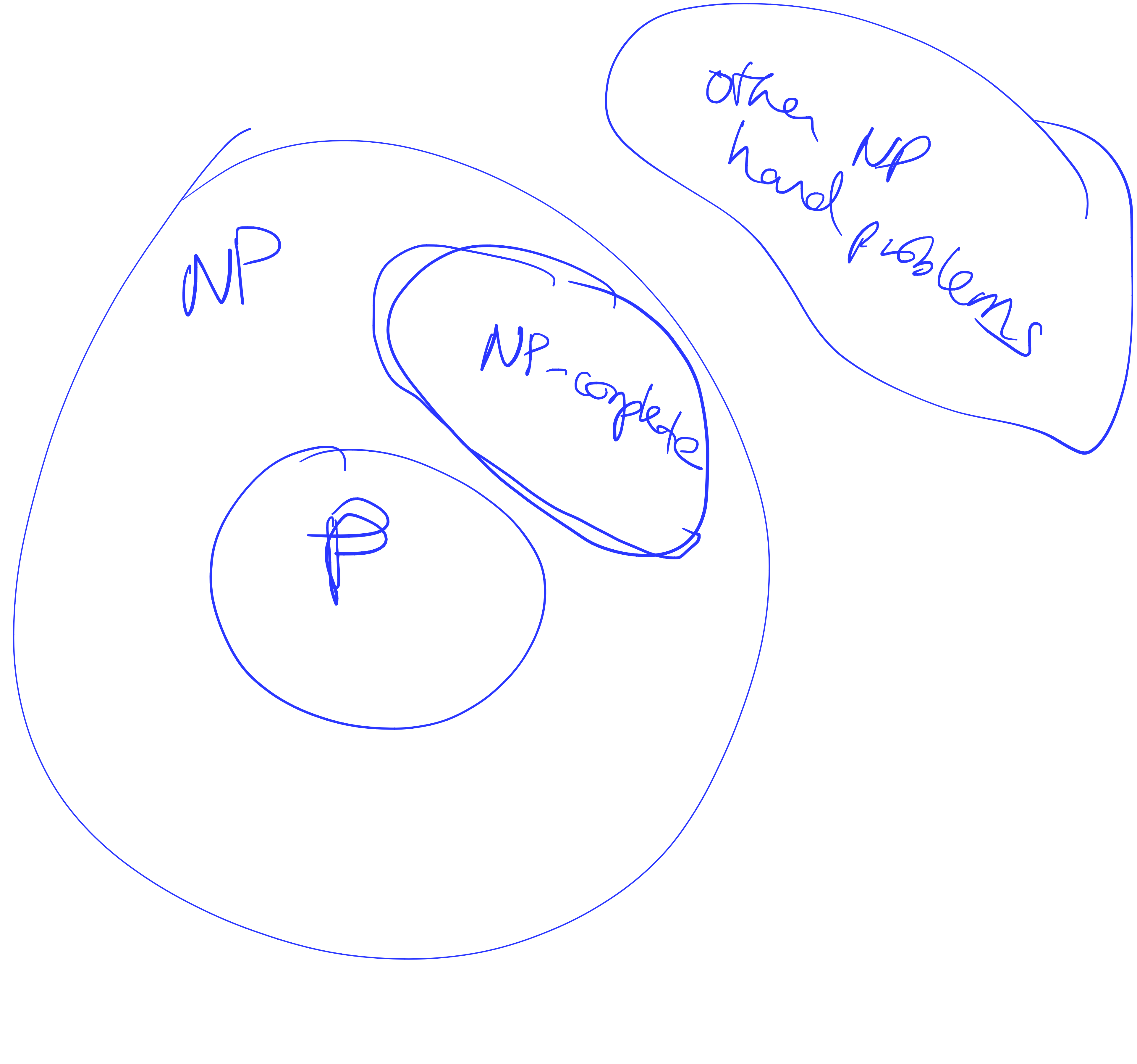
Need to show:

For any L that is NP complete, $\bar{L}$ is NP hard

$\equiv$ For any L that is NP-complete, if $\bar{L} \in P$ then <u>every</u> $L' \in NP$ is also in P

Example: say L is CLIQUE

show $\bar{L} = \overline{CLIQUE}$ is NP hard



NP
NP-complete
P
other NP hard problems

(5) K-minSAT

Recall KSAT = $\{ \phi \mid \phi$ is a KCNF formula and ϕ is satisfiable $\}$

$$\begin{array}{l} \\ \text{Example: } \phi = \end{array} \begin{array}{c} \scriptstyle 0\ \ 0\ \ 1\quad\ 0\ \ 1\quad\ 1\ \ 0\ \ 1\quad\ 1\ \ 1 \\ (x_2 \vee x_4 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3)(\bar{x}_1 \vee x_2 \vee \bar{x}_4)(\bar{x}_2 \vee \bar{x}_3) \end{array}$$

3SAT instance

Brute Force: Try all possible 2^n assignments (n = # vars)

x_1	x_2	x_3	x_4	
0	0	0	0	✓
0	0	0	1	
0	0	1	0	
0	0	1	1	
0	1	0	0	
0	1	0	1	
0	1	1	0	
0	1	1	1	
1	0	0	0	
1	0	0	1	
1	0	1	0	
1	0	1	1	
1	1	0	0	
1	1	0	1	
1	1	1	0	
1	1	1	1	

$\alpha = 1100$ satisfies ϕ so $\phi \in$ 3SAT

If $K=2$ then KMINSAT$(\phi)=1$ iff ϕ has a sat-ass with ≤ 2 1's

(5) K-minSAT : input to k-minSAT is a 3CNF formula

Recall KSAT $= \{ \phi \mid \phi$ is a 3CNF formula and ϕ is satisfiable $\}$

3-MINSAT $:= \{ \phi \mid \phi$ is a 3CNF formula and there is an assignment α with ≤ 3 1's that satisfies $\phi \}$

Example 1: $\phi = (x_2 \vee x_4 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3)(\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_4)(\bar{x}_2 \vee \bar{x}_3)$

$\alpha = 1\ 1\ 0\ 0$ satisfies ϕ and has ≤ 3 1's

so $\phi \in$ 3MINSAT

Example 2 $\phi' = (x_1 \vee \bar{x}_2)(x_2 \vee \bar{x}_4)(\bar{x}_1 \vee \bar{x}_2 \vee x_3)(x_4)(\bar{x}_4 \vee \bar{x}_5)$

$\alpha = 11110$ satisfies ϕ' so $\phi' \in$ 3SAT

But no assignment with ≤ 3 1's satisfies ϕ' so $\phi' \notin$ 3MINSAT

Examples of Languages in NP

1. Any $L \in P$ is also in NP

   Verifier V on input (w, c) : ignore c and just run polytime alg for L on input w.

2. CLIQUE $= \{ (g, k) \mid g$ is an undirected graph containing a size-k clique$\}$

   Verifier V on input $((g, k), s)$:

   - check that S encodes a subset $S \subseteq V$ of k vertices
   - For all pairs of vertices $i, j \in V'$ check if (i, j) is an edge in E (i.e., $(i, j) \in E$)

Examples of Languages in NP

(3) $K\text{-SAT} = \{ \phi \mid \phi$ is a satisfiable K-CNF formula $\}$

Input is a Boolean formula over $x_1, \dots, x_n$ in KCNF form.

KCNF form: $C_1 \wedge C_2 \wedge \dots \wedge C_m$

Example: $\phi = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_3 \vee x_4) \wedge (\bar{x}_2 \vee \bar{x}_1) \wedge (\bar{x}_2 \vee x_3)$

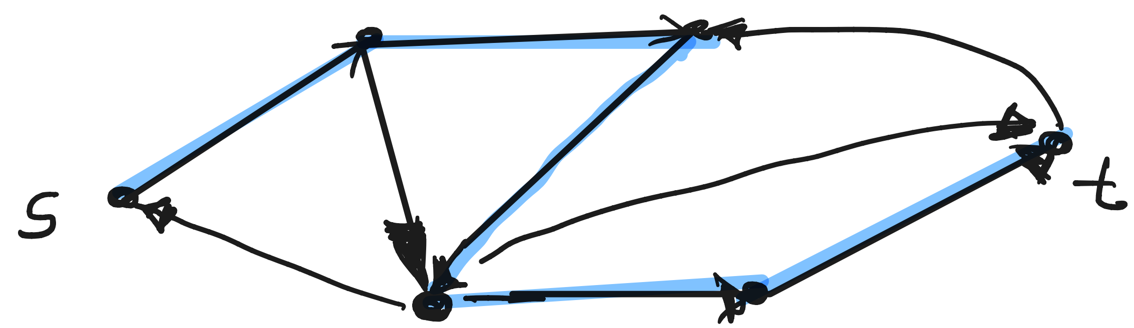
Verifier V on input (ϕ, α):

check that α is a Boolean satisfying assignment for ϕ. If yes $\rightarrow$ accept, otherwise $\rightarrow$ reject

Examples of Languages in NP

(4) HAMPATH = $\{(G, s, t) \mid G$ is a directed graph containing a Hamilton path (visits all vertices once) from s to $t\}$

Example

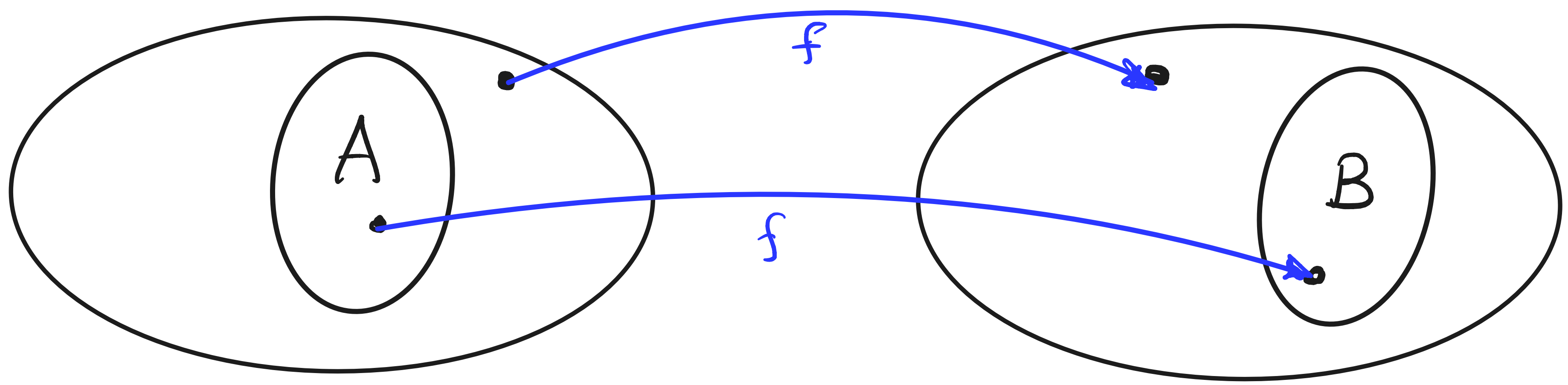


Verifier V on input $((G, s, t), p)$:

Check if p encodes a Hamiltonian path from s to t.
If yes → accept; otherwise → reject

NP - completeness

Definition Language A is polynomial-time (mapping) reducible to B (written $A \leq_p B$) if there is a polynomial-time computable function $f : \Sigma^* \to \Sigma^*$ such that $w \in A \iff f(w) \in B$

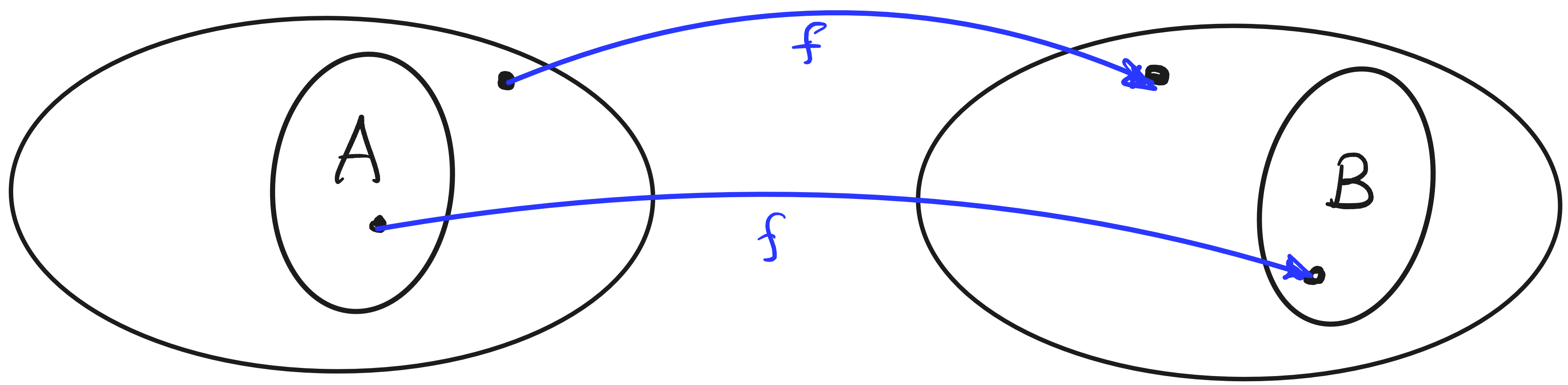


Definition

- A language $B \subseteq \{0,1\}^*$ is NP-hard if for <u>every</u> $A \in NP$ there is a polynomial time reduction from A to B $(A \leq_p B)$

NP - completeness

<ins>Definition</ins> Language A is polynomial-time (mapping) reducible to B (written $A \leq_p B$) if there is a polynomial-time computable function $f: \Sigma^* \to \Sigma^*$ such that $w \in A \Leftrightarrow f(w) \in B$

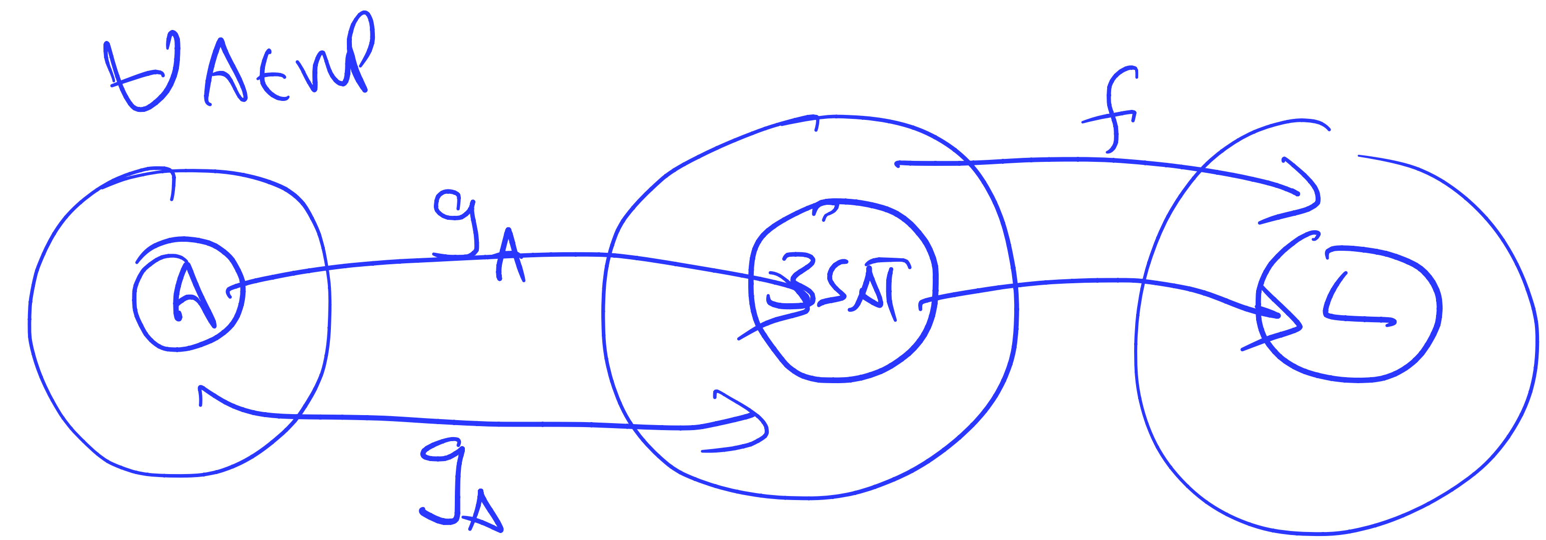


<ins>Definition</ins>

- A language $B \subseteq \{0,1\}^*$ is NP-hard if for <ins>every</ins> $A \in NP$ there is a polynomial time reduction from A to B ($A \leq_p B$)

- $B \subseteq \{0,1\}^*$ is NP-complete if: (i) B is in NP and
  (ii) B is NP-hard

NP - completeness



Cook-Levin theorem

For every $k \geq 3$ 3-SAT is NP-complete

(Proof Next week)

To show another language L is NP complete we just need to show:

(1) $L \in NP$

(2) Show $3SAT \leq_p L$

f

NP-completeness via Reductions

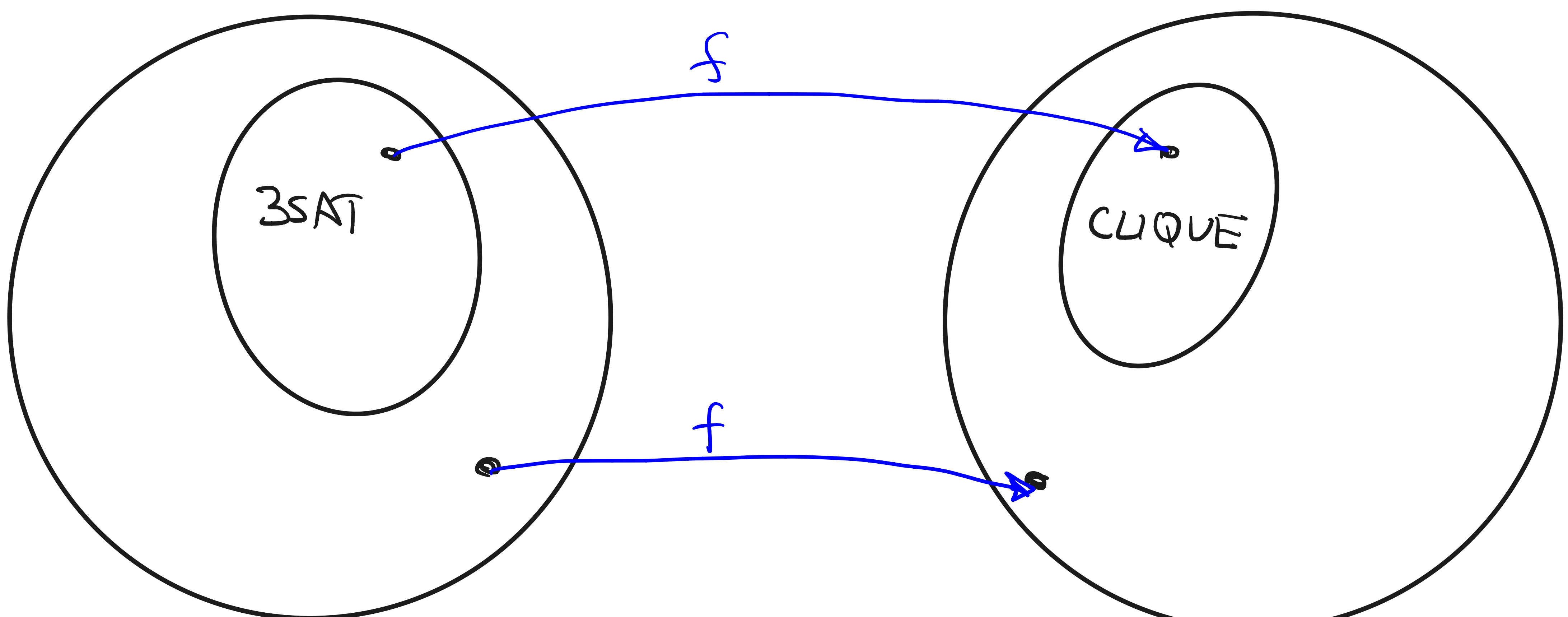
Clique = $\{ (G, k) \mid G$ contains a clique of size $\geq k \}$

Theorem CLIQUE is NP-complete

Proof (2 steps)

(i) CLIQUE is IN NP (we already showed this)

(ii) CLIQUE is NP-HARD: Need to show $3SAT \leq_p$ CLIQUE



3-SAT $\leq_p$ CLIQUE

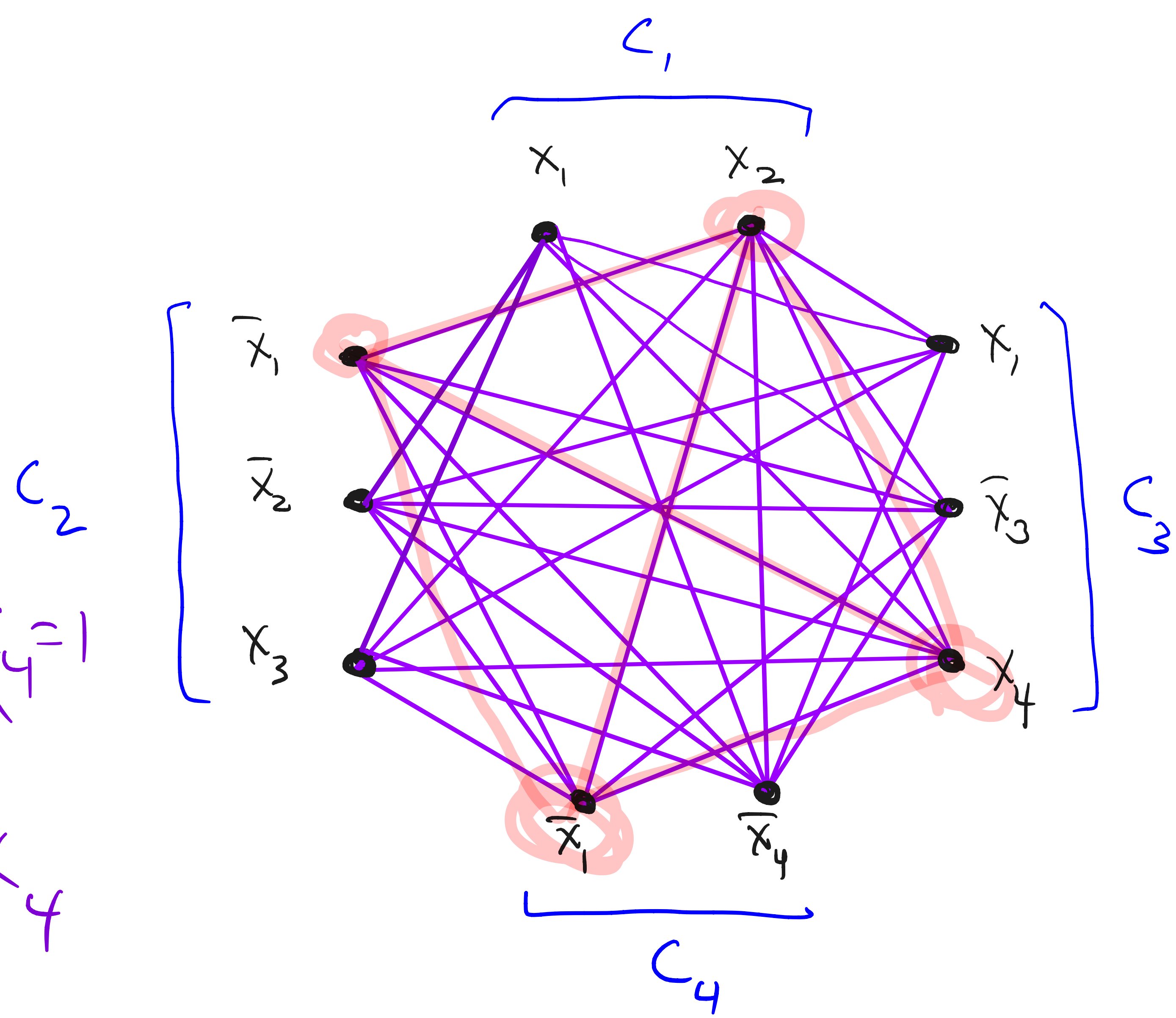
Example

Let $\phi = \underbrace{(x_1 \vee x_2)}_{C_1} \wedge \underbrace{(\bar{x}_1 \vee \bar{x}_2 \vee x_3)}_{C_2} \wedge \underbrace{(x_1 \vee \bar{x}_3 \vee x_4)}_{C_3} \wedge \underbrace{(\bar{x}_1 \vee \bar{x}_4)}_{C_4}$

m clauses
here $m = 4$

$f : \phi \longrightarrow (G_\phi, \ k = m)$

G_ϕ:



α:

$x_1 = 0 \quad x_2 = 1 \quad x_4 = 1$

$\uparrow \quad \uparrow \quad \uparrow$

$\bar{x}_1 \quad x_2 \quad x_4$

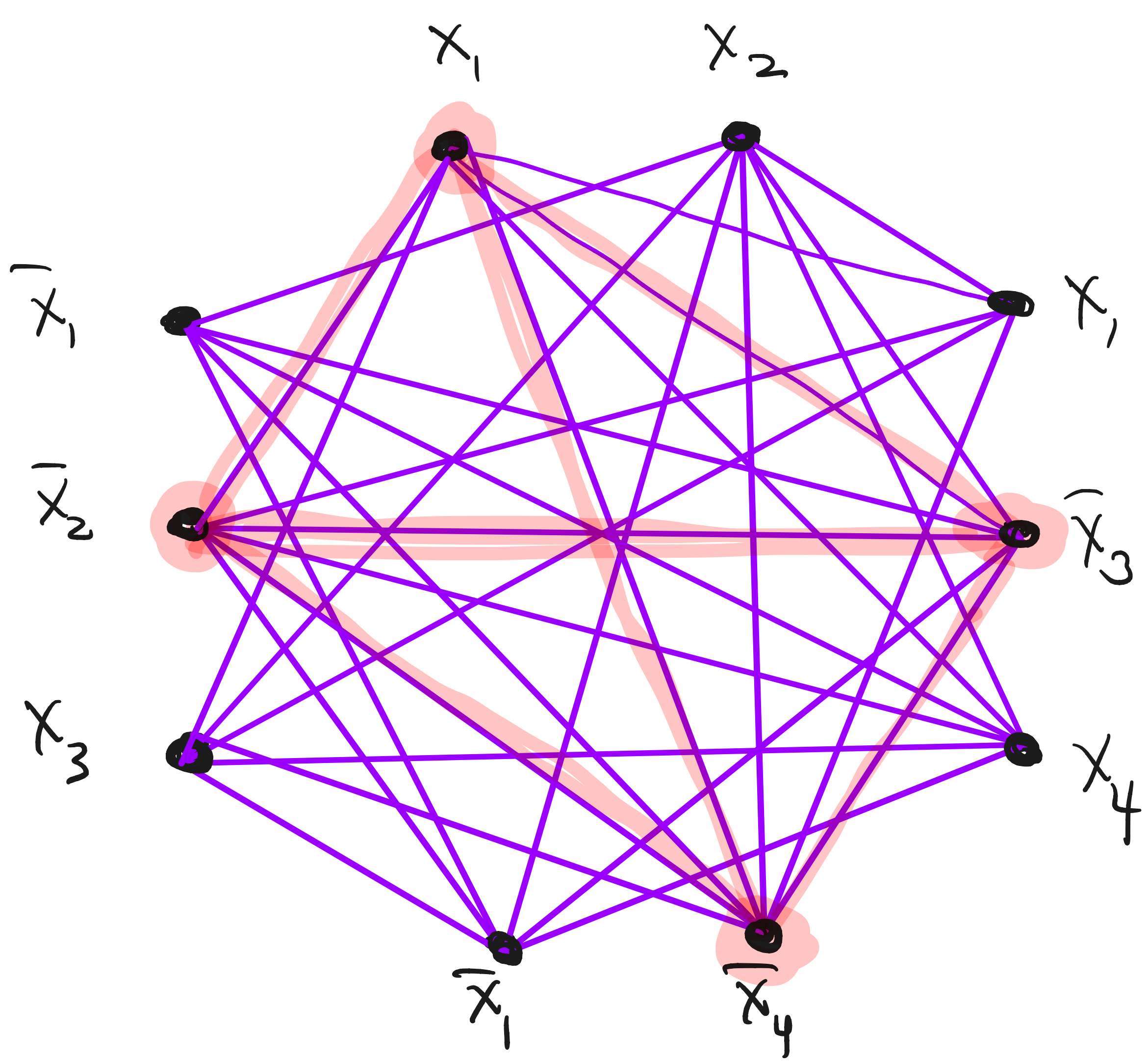
edge between 2 vertices if they are in different clause groups and if their associated literals are consistent

pick $k = m$ (the number of clauses in ϕ)

3-SAT $\leq_p$ CLIQUE

Example

Let $\phi = (x_1 \vee x_2) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_3) \wedge (x_1 \vee \bar{x}_3 \vee x_4) \wedge (\bar{x}_1 \vee \overline{x_4})$



edge between 2 vertices if they are in different clause groups and if their associated literals are consistent

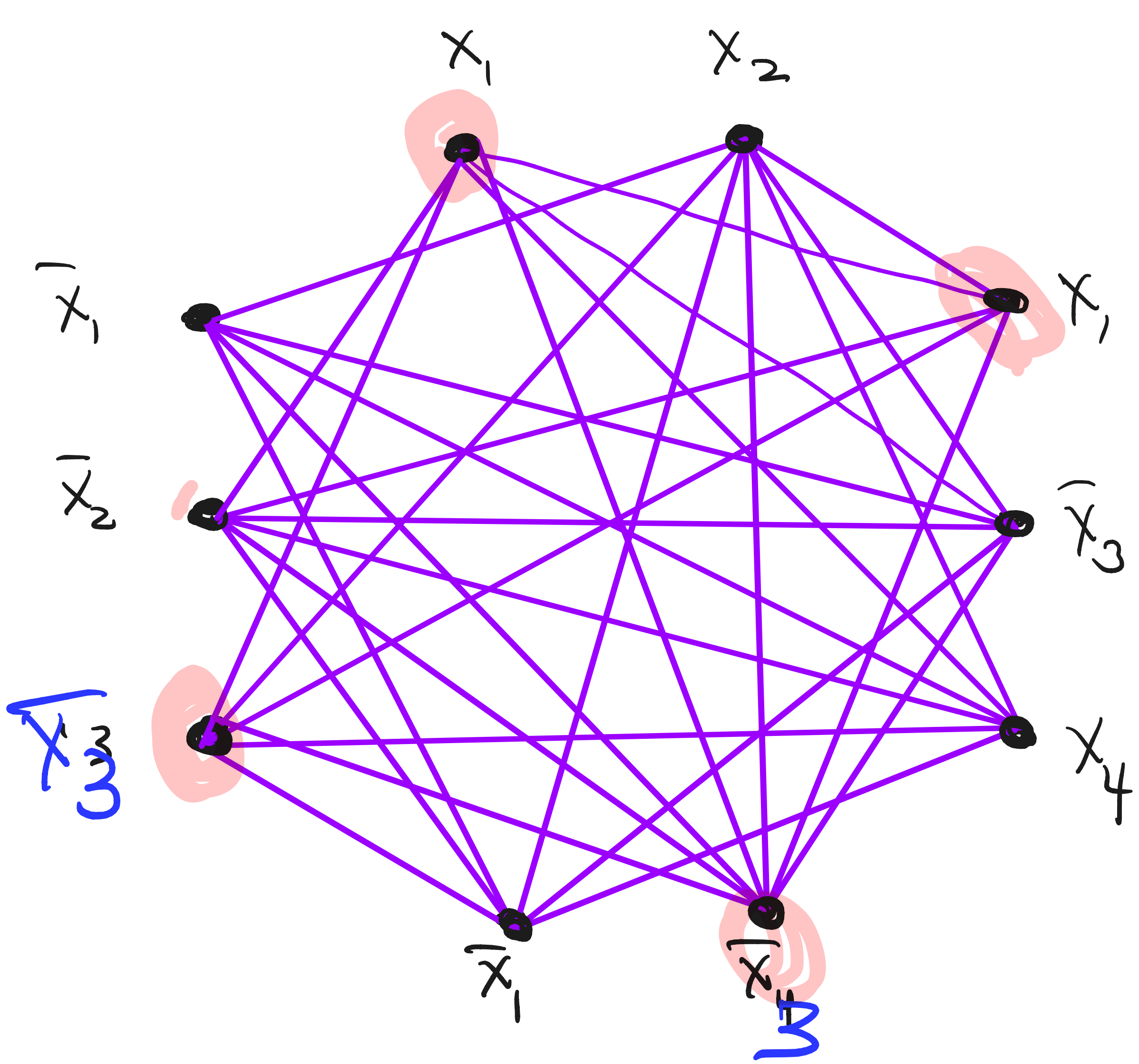
α : $x_1 = 1$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$ satisfies ϕ

corresponding 4-clique in red

3-SAT $\leq_p$ CLIQUE

Example

Let $\phi = (x_1 \vee x_2) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3) \wedge (x_1 \vee \bar{x}_3 \vee x_4) \wedge (x_1 \vee \bar{x}_3)$



edge between 2 vertices if they are in different clause groups and if their associated literals are consistent

$x_1 = 1 \quad x_3 = 0 \quad x_2 \ x_4$ set any way

We showed:

$f : \phi \longrightarrow (G_\phi, k)$

such that $\phi \in 3SAT \iff (G_\phi, k) \in CLIQUE$

Correctness: show

(1) If $\phi \in 3SAT$ (ϕ is satisfiable) $\longrightarrow$ $\overbrace{(G_\phi, k)}^{f(\phi)}$ has a k-clique

(2) If (G_ϕ, k) has a k-clique $\Rightarrow$ ϕ is sat.

NP-completeness via Reductions

Theorem HAMPATH is NP-complete

Proof

1. HAMPATH in NP (already did this)
2. We will show $3SAT \leq_p$ HAMPATH (and thus HAMPATH is NP-hard)

Let $\phi = (a_1 \vee b_1 \vee c_1) \wedge (a_2 \vee b_2 \vee c_2) \wedge \dots \wedge (a_m \vee b_m \vee c_m)$ each a_i, b_i, c_i is a literal.

$f : \phi \to (G_\phi, s, t)$

HAMPATH $= \{ (G, s, t) \mid G$ is a directed graph with a hamiltonian path (visits every vertex in G exactly once) from s to t $\}$

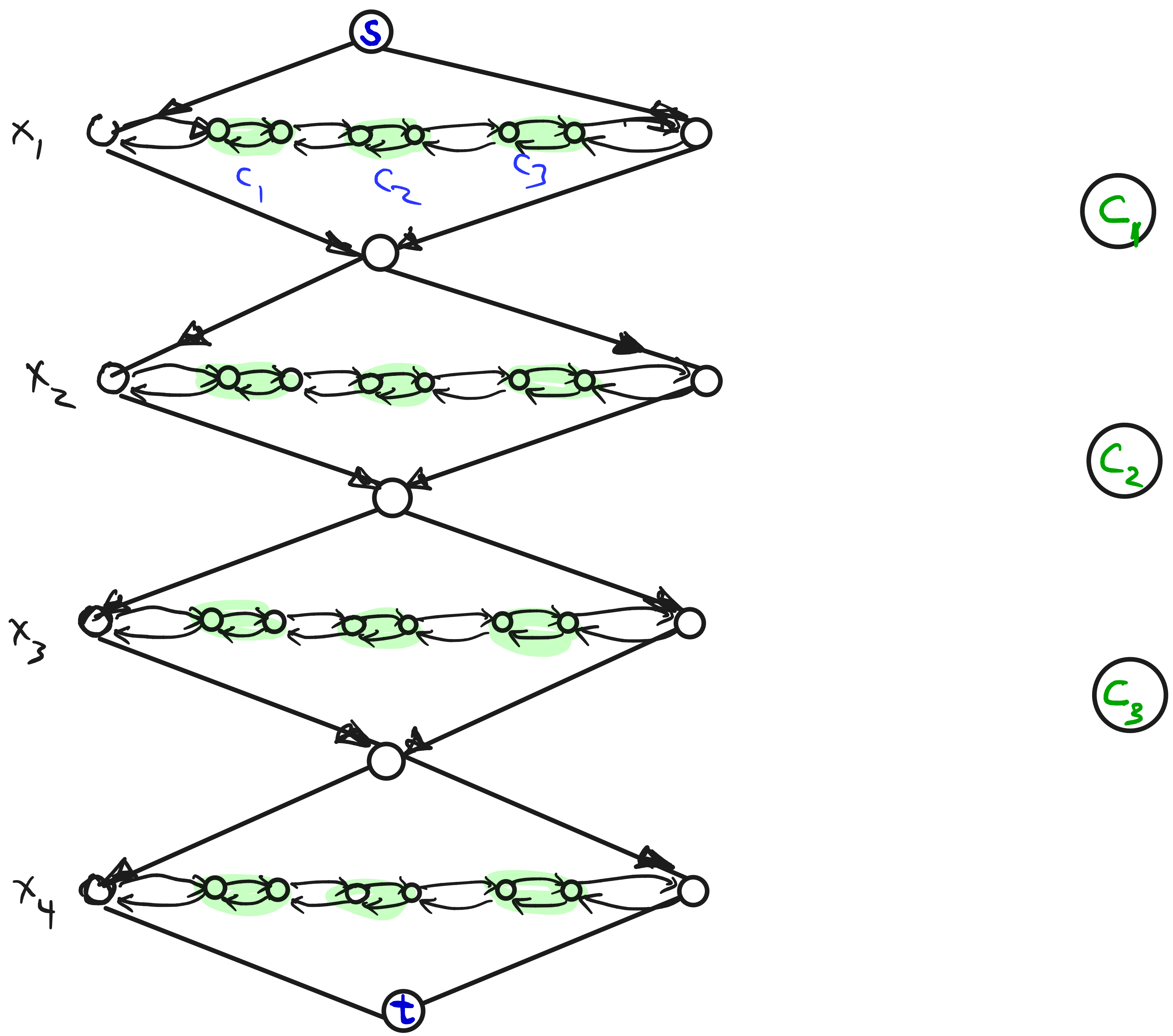
Let $\phi = \underbrace{(x_1 \vee x_3 \vee \bar{x}_4)}_{C_1} \wedge \underbrace{(x_2 \vee \bar{x}_3 \vee x_4)}_{C_2} \wedge \underbrace{(\bar{x}_1 \vee x_2 \vee x_3)}_{C_3}$

$n = \#\text{ vars} = 4$

$m = \#\text{ clauses} = 3$

$f : \phi \to (G_\phi, s, t)$

G_ϕ:



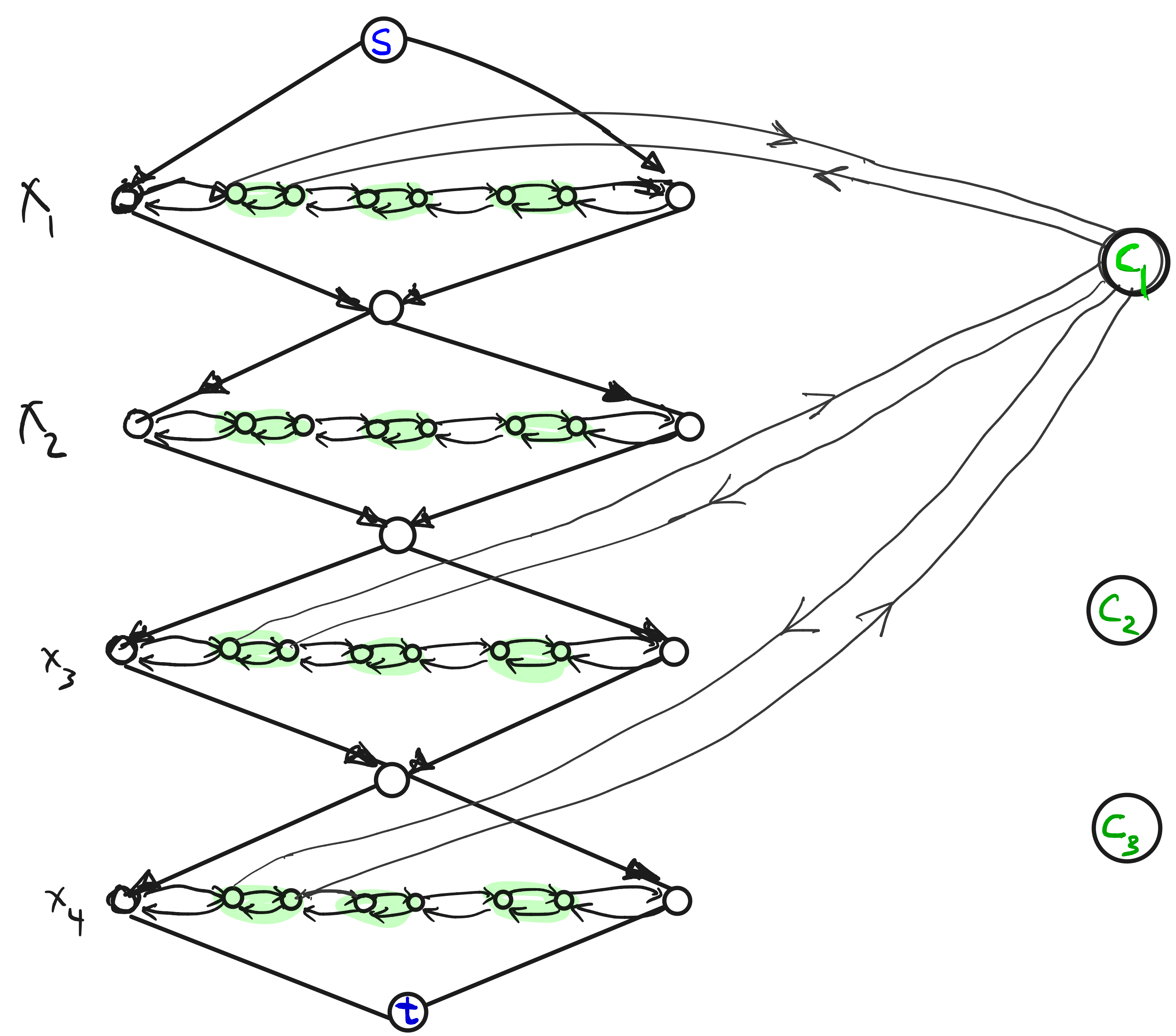
Let $\phi = \underbrace{(x_1 \vee x_3 \vee \bar{x}_4)}_{c_1} \wedge \underbrace{(x_2 \vee \bar{x}_3 \vee x_4)}_{c_2} \wedge \underbrace{(\bar{x}_1 \vee x_2 \vee x_3)}_{c_3}$

$n = \#\text{ vars} = 4$

$m = \#\text{ clauses} = 3$

$f : \phi \to (g_\phi, s, t)$

g_ϕ :



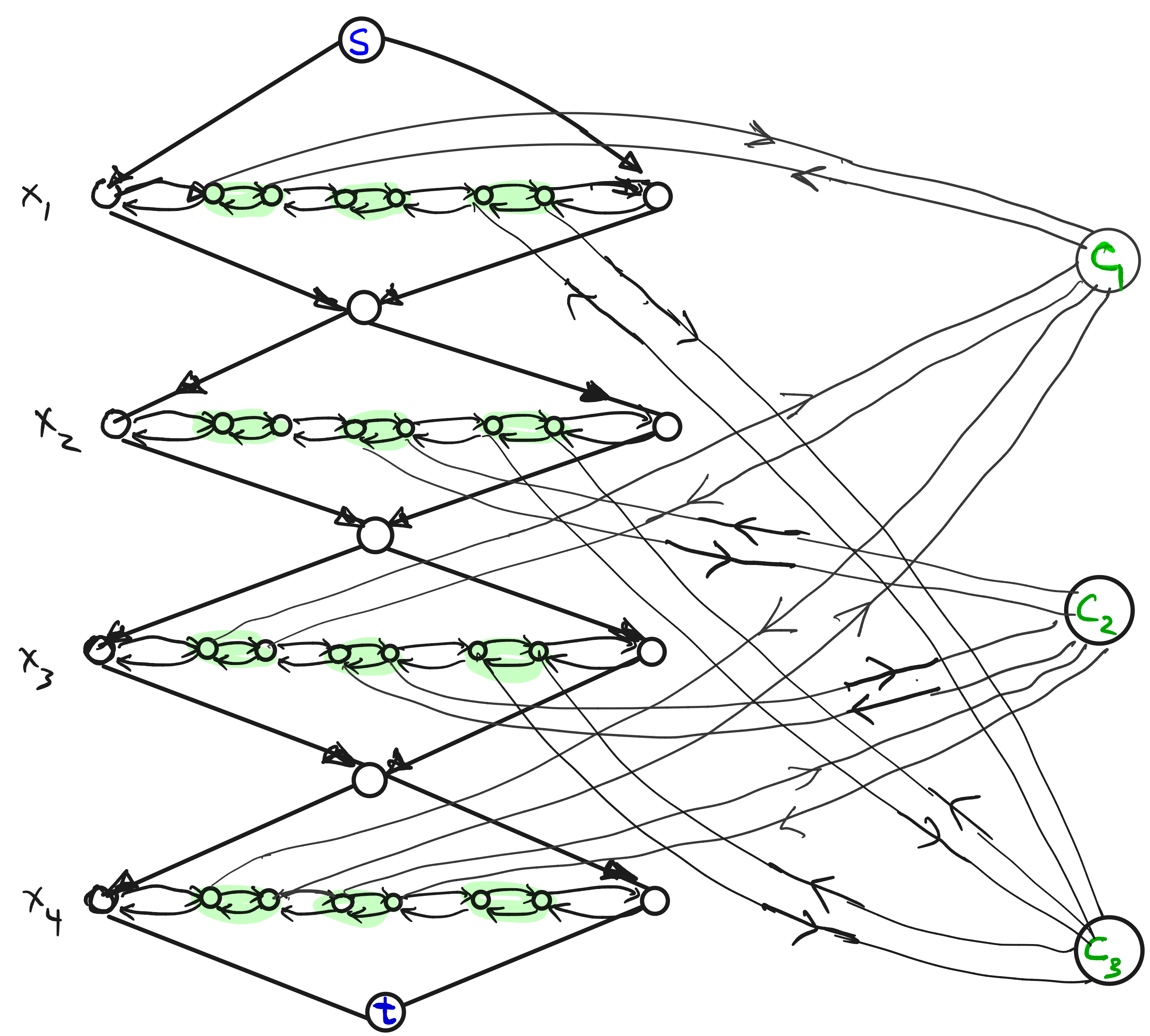
Let $\phi = \underbrace{(x_1 \vee x_3 \vee \bar{x}_4)}_{C_1} \wedge \underbrace{(x_2 \vee \bar{x}_3 \vee x_4)}_{C_2} \wedge \underbrace{(\bar{x}_1 \vee x_2 \vee x_3)}_{C_3}$

$n = \#\text{ vars} = 4$

$m = \#\text{ clauses} = 3$

$f : \phi \rightarrow (G_\phi, s, t)$

G_ϕ:



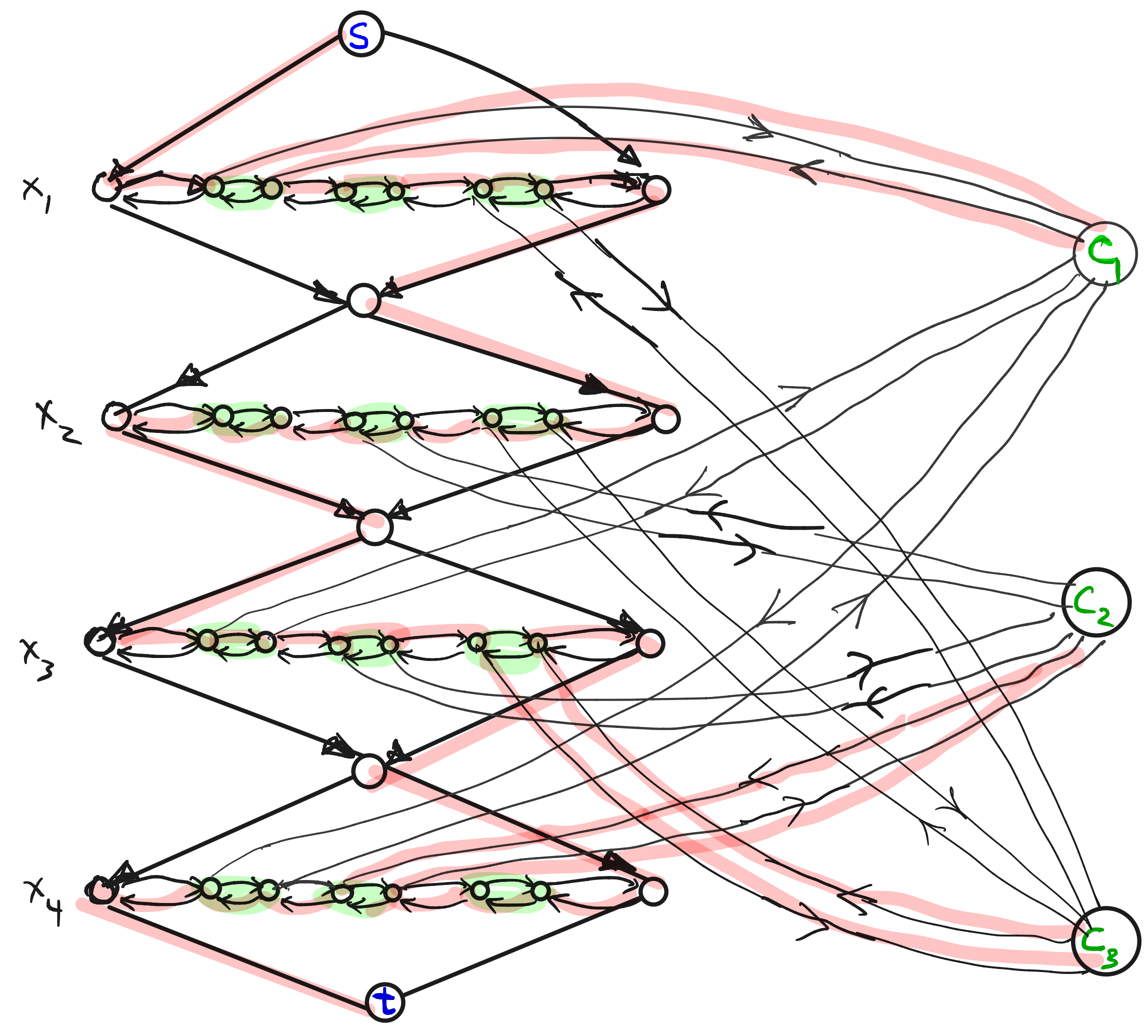
Let $\phi = \underbrace{(x_1 \vee x_3 \vee \bar{x}_4)}_{C_1} \wedge \underbrace{(x_2 \vee \bar{x}_3 \vee \bar{x}_4)}_{C_2} \wedge \underbrace{(\bar{x}_1 \vee x_2 \vee x_3)}_{C_3}$

$n = \#\text{ vars} = 4$

$m = \#\text{ clauses} = 3$

$f : \phi \rightarrow (g_\phi, s, t)$

g_ϕ :



Claim ϕ is satisfiable iff g_ϕ has a Hamiltonian path from s to t

α : $\underbrace{x_1 = 1}_{C_1}$ $x_2 = 0$ $\underbrace{x_3 = 1}_{C_3}$ $\underbrace{x_4 = 0}_{C_2}$