Lecture 17

HW3 out (Due Nov 20)

Recap from Last Week: Proving there are Languages not r.e.

Idea: show there are more languages than r.e. languages.
To compare sizes of infinite sets we use the Notion of countable/uncountable.

DefN A set S is countable if there is a 1-to-1 mapping from $S \to \mathbb{N}$

Example $\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, \ldots\}$

...

So
$f(0) = 0$
$f(1) = 1$
$f(2) = -1$
$f(3) = 2$
$f(4) = -2$
$f(5) = 3$
$f(6) = -3$
$\vdots$

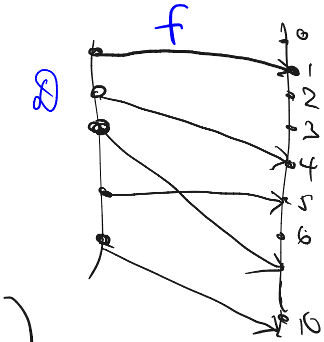
Define $f(i) = \begin{cases} -\frac{i}{2} & \text{if } i \text{ is even} \\ \frac{i+1}{2} & \text{if } i \text{ is odd} \end{cases}$

maps $\mathbb{N} \xrightarrow[\text{onto}]{}$ integers

A mapping $f : \overset{\text{domain}}{\mathcal{D}} \longrightarrow \overset{\text{range}}{\mathcal{R}}$ is 1-1 if

for every element $y \in \mathcal{R}$, there is at most one $x \in \mathcal{D}$ that maps to y

$(\forall y \in \mathcal{R} \ \exists$ at most one $x \in \mathcal{D}$ s.t. $f(x) = y)$

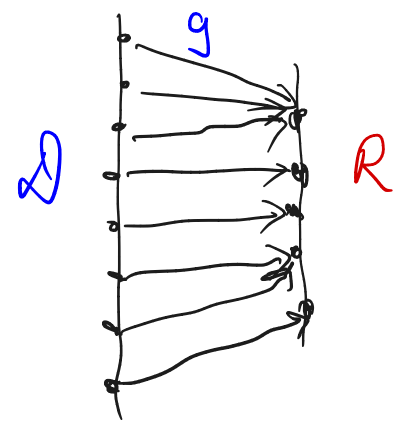


⟸ This map f is 1-to-1

A mapping $g : \mathcal{D} \longrightarrow \mathcal{R}$ is onto if

for every $y \in \mathcal{R}$ there is <u>at least</u> one $x \in \mathcal{D}$ that maps to y

$(\forall y \in \mathcal{R} \ \exists$ at least one $x \in \mathcal{D}$ s.t. $f(x) = y)$

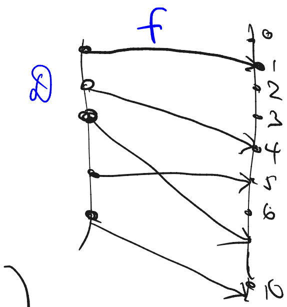


this map g is onto

A mapping $f: \underbrace{D}_{\text{domain}} \rightarrow \underbrace{R}_{\text{range}}$ is 1-1 if

for every element $y \in R$, there is at most one $x \in D$ that maps to y

$(\forall y \in R \ \exists$ at most one $x \in D$ s.t. $f(x) = y)$



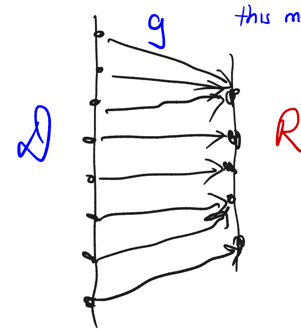
← This map f is 1-to-1

A mapping $g: D \rightarrow R$ is onto if

for every $y \in R$ there is at least one $x \in D$ that maps to y

$(\forall y \in R \ \exists$ at least one $x \in D$ s.t. $f(x) = y)$

this map g is onto



HW3 Question: Prove for any infinite set S

$\exists$ a function $f: S \rightarrow \mathbb{N}$ that is 1-1 if and only if

$\exists$ a function $g: \mathbb{N} \rightarrow S$ that is onto

Idea: show there are way more languages than r.e. languages.

To compare sizes of infinite sets we use the Notion of countable vs uncountable.

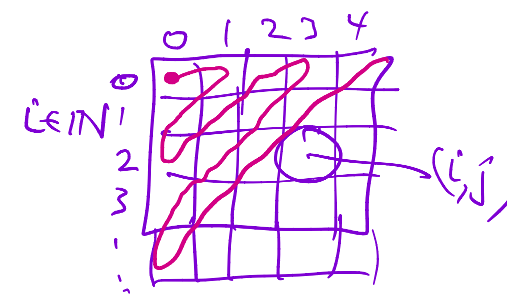
DefN A set S is countable if there is a 1-1 mapping from $S \to \mathbb{N}$

More Examples of countable Sets

① Any finite set S is countable

② The set of all pairs $(i,j) \in \mathbb{N} \times \mathbb{N}$

③ $\mathbb{Q} = \{ \frac{m}{n} \mid m, n \in \mathbb{N} \}$



Lemma (HW3 problem)

S countable $\Rightarrow \exists$ an onto mapping $g : \mathbb{N} \to S$

DefN A set S is countable if there is a 1-1 mapping from $S \to \mathbb{N}$

How to show that some (infinite) set is Not countable ?

Proof Diagonalization argument

(similar to Cantor's argument showing that the set of all real numbers is uncountable by showing there is NO 1-1 map from $\mathbb{R} \Rightarrow \mathbb{N}$)

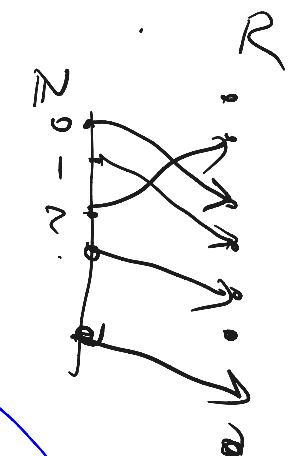
reals

Example The set of all real numbers in $[0,1]$ is not countable.

Suppose (for contradiction) $\exists$ a 1-1 mapping g from $\mathbb{R} \to \mathbb{N}$

By HW3 problem this implies $\exists$ an onto mapping $f : \mathbb{N} \to \mathbb{R}$

$i=0$	.0	1	7	4	3	9	8	...			$\leftarrow f(0)$
$i=1$	.0	0	2	5	6	9	0	2	2	...	$\leftarrow f(1)$
$i=2$	.1	1	8	4	0	0	...				$\vdots$
$i=3$	.1	1	2	3	...						
$\vdots$											



Construct real number $x = .x_0 x_1 x_2 \ldots$ such that $x_i \neq f(i)_i$

e.g. Let $x_i = \begin{cases} 0 & \text{if } f(i)_i \neq 0 \\ 1 & \text{if } f(i)_i = 0 \end{cases}$

$x = .1100\ldots$

Since x not any row of table (by construction), f is not onto mapping. #

Example 2 The set of all languages over $\{0,1\}^*$ is not countable.

Suppose (for contradiction) $\exists$ an onto mapping $f: \mathbb{N} \to \{L \mid L \subseteq \{0,1\}^*\}$

enumeration of all $w \in \{0,1\}^*$

	ε	0	1	00	01	10	11	000	001	010	....	
$i=0$	$\cancel{0}\to 1$	1	1	0	0	1	1	1	1	1	..	$\leftarrow f(0)$
$i=1$	1	$\cancel{0}\to 1$	1	1	1	0	1	1	0	0	...	$\leftarrow f(1)$
$i=2$	1	0	$\cancel{1}\to 0$	0	0	...						$\leftarrow f(2)$
$\vdots$	1	1	1	$\cancel{0}\to 1$	0	.						$\vdots$

Row i of table is the i^{th} language (according to the ordering given by f)

Let $L = \{ w \mid w \text{ is the } i^{th} \text{ string over } \{0,1\}^* \text{ and } f(i)_i = 0 \}$

L is a language not in range of f. #

Example 2 The set of all languages over $\{0,1\}^*$ is Not countable.

Suppose (for contradiction) $\exists$ an onto mapping $f: \mathbb{N} \to \{ L \mid L \subseteq \{0,1\}^* \}$

enumeration of all $w \in \{0,1\}^*$

	ε	0	1	00	01	10	11	000	001	010	....	
$i=0$	$\cancel{0} \nearrow 1$	1	1	0	0	1	1	1	1	1	..	$\leftarrow f(0)$
$i=1$	1	$\cancel{0} \nearrow 1$	1	1	1	0	1	1	0	0	...	$\leftarrow f(1)$
$i=2$	1	0	$\cancel{1} \nearrow 0$	0	0	...						$\leftarrow f(2)$
$\vdots$	1	1	1	$\cancel{0} \nearrow 1$	0							$\vdots$

$L = \{ \varepsilon, 0, 00, \ldots \}$

Row i of table is the i^{th} language (according to the ordering given by f)

Let $L = \{ w \mid w$ is the i^{th} string over $\{0,1\}^*$ and $f(i)_i = 0 \}$

L is a language Not in range of f. #

$L = b_0 b_1 b_2 \ldots\ldots\ldots$ $b_i = \{ 1$ iff $f(i)_i = 0 \}$

what goes wrong if I try to show the set of integers $\{\ldots -3, -2, -1, 0, 1, \ldots\}$ is uncountable?

assume fsoc $\exists f : \mathbb{N} \xrightarrow[\text{onto}]{} \mathbb{Z} \leftarrow$ integers

0	$\boxed{-3}$
1	-7
2	10
3	11
4	100
	4

← -11 1

-111 11

← 1010 011

← f(0)

← f(1)

⋮

Theorem There exists languages over $\{0,1\}^*$ that are not r.e.

Proof 1 Let $\Sigma = \{0,1\}$

- the set of all Turing-recognizable languages is countable
- But on the other hand by Example 2, the set of all languages over $\{0,1\}^*$ is uncountable
- Since $\{\text{r.e. languages}\} \subseteq \text{ALL } L \subseteq \Sigma^*$

$\exists$ a language $L \subseteq \Sigma^*$ that is not r.e.

Theorem There exists a Language $L \subseteq \{0,1\}^*$ that is not r.e. (recognizable)

Pf 2 (more explicit)

Fix an enumeration of all TMs over $\{0,1\}$ using our encoding of TMs

$M_1, M_2, M_3, \ldots$

order lexicographically by their encodings (so $\langle M_1 \rangle < \langle M_2 \rangle < \ldots$)

$$\text{Entry}\left(M_i, \langle M_j \rangle\right) = \begin{cases} 1 & \text{if } M_i \text{ accepts } \langle M_j \rangle \\ 0 & \text{otherwise} \end{cases}$$

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$
M_1	0	1	0	0	$\cdots$
M_2	1	0	1	1	$\cdots$
M_3	0	1	1	1	
M_4	0	1	0	1	
$\vdots$					

1 iff $M_3(\langle M_4 \rangle)$ accepts

Let $\tilde{D} = \{ \langle M \rangle \mid \langle M \rangle$ encodes a TM, M and $M(\langle M \rangle)$ accepts $\}$

Define $D = \{ \langle M \rangle \mid \langle M \rangle \text{ encodes TM } M, \text{ and } M \text{ does not accept } \langle M \rangle \}$

D is called the "diagonal language"

since it does the opposite of what was on the diagonal

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$
M_1	1	1	0	0	$\cdots$
M_2	1	1	1	1	$\cdots$
M_3	0	1	0	1	
M_4	0	1	0	0	
$\vdots$					

$$D(\langle M_i \rangle) = \begin{cases} 0 & \text{if } M_i(\langle M_i \rangle) \text{ accepts} \\ 1 & \text{otherwise} \end{cases}$$

$$\tilde{D}(\langle M \rangle) = \begin{cases} 1 & \text{if } M(\langle M \rangle) \text{ accepts} \\ 0 & \text{otherwise} \end{cases}$$

<u>Theorem</u> There exists a Language $L \subseteq \{0,1\}^*$ that is not r.e. (recognizable)

<u>Pf</u> (diagonalization)

Fix an enumeration of all TMs over $\{0,1\}$ using our encoding of TMs

$M_1, M_2, M_3, \ldots$

order lexicographically by their encodings (so $\langle M_1 \rangle < \langle M_2 \rangle < \ldots$)

Define $D = \{ \langle M \rangle \mid \langle M \rangle$ encodes TM M, and M on input $\langle M \rangle$ does not halt and accept $\}$

the Diagonal Language

<u>Claim</u> D is not r.e. (recognizable)

<u>Pf</u> By construction, for all TMs M_i over Σ,

$\mathcal{L}(M_i) \neq D$ since $\langle M_i \rangle \in D$ if and only if $\langle M_i \rangle \notin \mathcal{L}(M_i)$

Recall $\tilde{D} = \{ \langle M \rangle \mid M(\langle M \rangle) \text{ accepts} \}$

Note: $\tilde{D}$ is similar to A_{TM}

$$A_{TM} = \{ \langle M, w \rangle \mid M \text{ accepts } w \}$$

Claim $\tilde{D}$ is recognizable / r.e.

Pf : TM for $\tilde{D}$ on input $\langle M \rangle$

- check to see if input is legal encoding of a TM
  if not, reject
- otherwise run M on $\langle M \rangle$:
  If simulation halts and accepts $\rightarrow$ halt + accept

A note on D, and $\tilde{D}$ vs $\overline{D}$

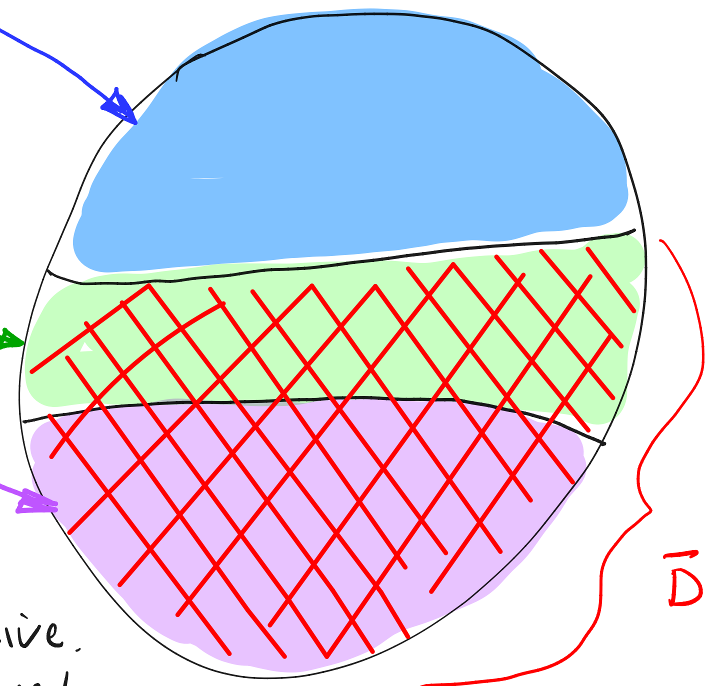
all strings $w \in \{0,1\}^*$

$D = \{ \langle M \rangle \mid M(\langle M \rangle) \text{ does not accept} \}$

$\overline{D}$ = complement of D

$\text{Illegal} = \{ w \mid w \text{ does not encode a legal TM} \}$

$\tilde{D} = \{ \langle M \rangle \mid M(\langle M \rangle) \text{ accepts} \}$



Note: $\tilde{D}$ recursive iff $\overline{D}$ recursive

since $\overline{D} = \tilde{D} \cup \text{Illegal}$ and Illegal is recursive.

Thus we will sometimes write $\tilde{D}$ as complement of D (since distinction not that important)

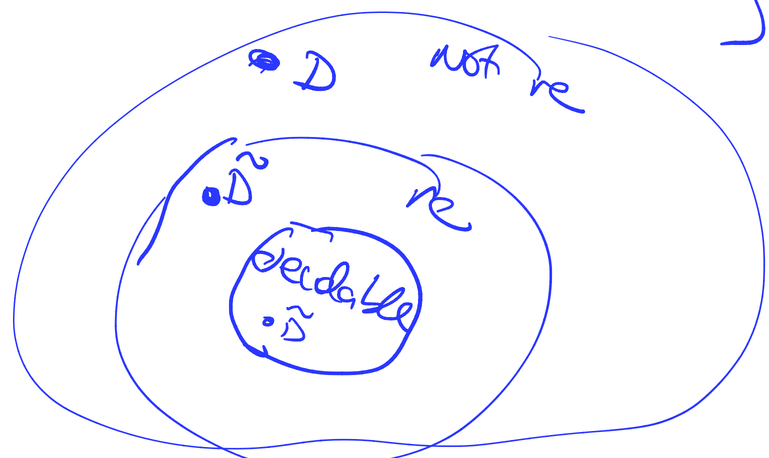
Thus we have shown: D is not r.e.

$\tilde{D}$ is r.e.

Question Is $\tilde{D}$ recursive (decidable)?

Remember by closure property we have:

$\forall L$ [L is decidable iff $\bar{L}$ is decidable }



Thus we have shown: $\tilde{D}$ is r.e., not recursive

D is not r.e. ← by diagonalization

Question Is $\tilde{D}$ recursive (decidable)?

No! Since decidable languages are closed under complement, $\bar{D}$ is not decidable.

In more detail:

$D = \{ \langle M \rangle \mid M$ codes TM M and $M(\langle M \rangle)$ does not accept

$\bar{D} = \underbrace{\{ w \mid w \text{ does not code a TM} \}}_{\text{decidable}} \cup \underbrace{\{ \langle M \rangle \mid \langle M \rangle \text{ codes } M \text{ and } M(\langle M \rangle) \text{ accepts} \}}_{\tilde{D}}$

$\therefore$ $\bar{D}$ not decidable since if it were, D would be decidable.

$\therefore$ $\tilde{D}$ not decidable since if it were $\bar{D}$ would be decidable

Other Languages that are Not Decidable

- Recall $A_{TM} = \{\langle M, x\rangle \mid w = \langle M\rangle$ encodes some TM M, and M accepts $x\}$
- We saw That A_{TM} is r.e./recognizable.

Pf that A_{TM} is not decidable:

Assume for sake of contradiction there is a decider N for A_{TM}.

We will use M to construct a decider N' for $\tilde{D}$:

N':

On input $\langle M\rangle$:
    check if input is a legal encoding of a TM. If not reject
    Otherwise Run N on $\langle M, \langle M\rangle\rangle$
        If N accepts $\longrightarrow$ reject
        If N rejects $\longrightarrow$ accept

Since N always halts, N' always halts.
Also N' accepts $\tilde{D}$. Contradiction since $\tilde{D}$ is not decidable

$\therefore$ A_{TM} is not decidable