Pumping Lemma and PDA Review Session

October 5, 2023

1 Definitions

1.1 Pumping Lemma

Definition:

If L is a regular language, then there exists a number p, such that for all strings w where $w \in L$ and $|w| \geq p$, there exists a way to divide w into three pieces, $w = xyz$, satisfying the conditions

1. $|y| > 0$
2. $|xy| \leq p$
3. for each $i \geq 0, xy^iz \in L$

NOTE: There are non-regular languages that also satisfy the pumping lemma, the pumping lemma must be applied carefully to prove non-regularity. In particular, the contrapositive of the pumping lemma is used to prove non-regularity of a language.

Template: For a language L, if for all p, there exists a string w where $w \in L$ and $|w| \geq p$, such that every way to divide w into three pieces $w = xyz$ satisfying $|y| > 0$ and $|xy| \leq p$, there exists an $i \geq 0$ such that $xy^iz \notin L$, then L is not regular.

1.2 PDA

Definition: A PDA is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where Q, Σ, Γ, and F are finite sets, and

1. Q is the set of states
2. Σ is the input alphabet
3. Γ is the stack alphabet
4. $\delta : Q \times \{\Sigma \cup \epsilon\} \times \{\Gamma \cup \epsilon\} \rightarrow \mathcal{P}(Q \times \{\Gamma \cup \epsilon\})$
5. $q_0 \in Q$ is the start state

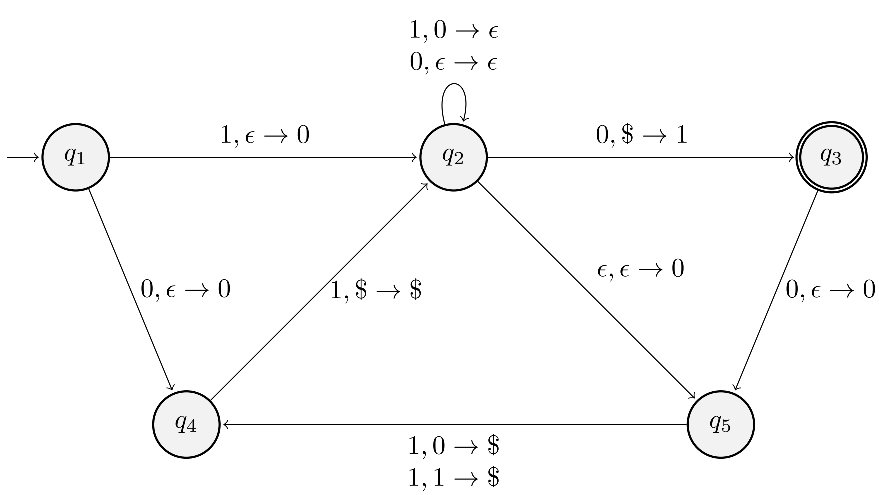
6. F is the set of accepting states

A PDA M accepts a word w if w can be written as $w = w_1 w_2 w_3 \cdots w_n$ where $w_i \in \{\Sigma \cup \epsilon\}$ and there is a sequence of states $r_0, r_1, r_2, \cdots, r_n \in Q$ and a sequence of stack content strings $s_0, s_1, s_2, \cdots, s_n \in \{\Gamma \cup \epsilon\}^*$ that satisfy the following conditions:

1. $r_0 = q_0$ and $s_0 = \epsilon$ indicating that M starts on the start state with an empty stack.
2. M makes a valid transition at each step: for $i = 0, 1, \cdots, n-1$, $(r_{i+1}, b) \in \delta(r_i, w_{i+1}, a)$, where $s_i = at$ and $s_{i+1} = bt$ for some $a, b \in \{\Gamma \cup \epsilon\}^*$ and $t \in \Gamma$.
3. $r_n \in F$ indicating that the computation ends on an accepting state.

2 Practice Problems

1. Use the pumping lemma to prove the following languages are not regular:
   (a) $L = \{0^k 1 w 0^k | k \geq 1 \text{ and } w \in \{0,1\}^*\}$
   (b) $L = \{0^k | k \text{ is a power of } 2\}$
   (c) $L = \{0^i 1^j | i > j\}$
   (d) $L = \{w \in \{0,1\}^* | \text{w has equal number of 0's and 1's}\}$
2. Consider the language $L = \{a^i b^j c^k | i, j, k \geq 0 \text{ and if } i = 1 \text{ then } j = k\}$
   (a) Show that L satisfies the pumping lemma
   (b) Show that L is non-regular
3. Run the strings 10, 010, and 1101 on the PDA below:



4. Construct PDA's for the following languages:

   (a) $L = \{w \in \{0,1\}^* | \, |w| \text{ is odd and the middle bit of w is a } 1\}$

   (b) The set of all strings over $\{0, 1, [,], (,)\}$ where parenthesis and brackets are properly matched.

   (c) The set of all strings over $\{0, 1\}^*$ that do not contain the substring 001, or have more 1's than 0's.