CS Theory Fall 2022 Handout 6b: Context Free Languages with SOLUTIONS

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October 20, 2022

1 CFG/CFL Overview

1.1 Key terms / facts

(a) A *context-free grammar (CFG)* is represented as a 4-tuple (V, Σ, R, S), where

- V is a set of *variables* or *nonterminals*
- Σ is a set of alphabet symbols or *terminals*
- R is a set of rules of the form $A \to \alpha$, where $A \in V$ and $\alpha \in (V \cup \Sigma)^*$
- $S \in V$ is the designated start variable.

(b) Let $G = (V, \Sigma, R, S)$ be a grammar. Let $\alpha A \beta$ be a string of variables and alphabet symbols, or $\alpha, \beta \in (V \cup \Sigma)^*$ and $A \in V$. If there is a rule in R of the form $A \to \gamma$, where $\gamma \in (V \cup \Sigma)^*$, we write

$$\alpha A \beta \Longrightarrow \alpha \gamma \beta$$

and say $\alpha A\beta$ yields $\alpha\gamma\beta$. If there is a finite sequence of w_i so that

$$w \Longrightarrow w_1 \Longrightarrow \cdots \Longrightarrow w_n \Longrightarrow z$$

we write $w \stackrel{*}{\Longrightarrow} z$ and say w derives z.

(c) The language generated by a CFG $G = (V, \Sigma, R, S)$ is defined as

$$L(G) = \{w \in \Sigma^* \mid S \stackrel{*}{\Longrightarrow} w\}$$

If for a language L there exists a CFG G satisfying $L(G) = G$, we say L is a *context-free language (CFL)*.

(d) **Fact:** the set of regular languages is a proper subset of the set of context-free languages. In class, we saw one proof that regular languages are context-free, by transforming regular expressions to equivalent CFGs. A different proof follows from transforming DFAs to CFGs – we will show a proof below. A third proof follows from transforming NFAs to PDAs (which is an immediate transformation, as NFAs can be viewed as a special case of PDAs) – we will also mention this below. On the other hand, there are non-regular, context-free languages, such as $\{a^i b^i : \forall i \geq 0\}$. Thus, the regular languages are a proper subset of CFL's.

(e) A CFG (Q, Σ, V, R) is *right-linear* if every rule in R is of the form $A \to \epsilon$ or $A \to aB$ where $A, B \in V, a \in \Sigma$. We stated the theorem that a language is regular if and only if it is generated by a right-linear CFG. Below, we will prove one direction of this theorem (we will prove that every regular language has a corresponding right-linear CFG).

(f) We have a *leftmost derivation* if we replace the leftmost variable with one of its production bodies in every derivation step. For a given grammar G and a string $w \in L(G)$, each leftmost derivation of w corresponds to a unique parse tree. Similarly, each parse tree for w corresponds to a unique leftmost derivation for w.

(g) A CFG is *ambiguous* if we can find a string w in Σ^* having two different parse trees with S as root that both generate w. This also means that the string w has two distinct leftmost derivations. A CFG is *unambiguous* otherwise.

(h) A CFL is *inherently ambiguous* if all of its grammars are ambiguous. In other words, no matter how you formulate a grammar for the language, there will always be some string that has two different leftmost derivations.

(i) Remember DFA's and NFA's? Well, CFG's come with their own kinds of automata called *pushdown automata (PDA)*. We define these further down. Simply put, they are NFA's associated with a stack. **A language is context free if and only if there exists a PDA which recognizes it.**

(j) Remember the pumping lemma for natural languages? Well, CFL's have their own pumping lemma, often called the *tandem pumping lemma*. The lemma states that **if L is a CFL, L has some associated pumping length p such that** $\forall w \in L \ |w| \geq p, \ \exists u, v, x, y, z$

i) $w = uvxyz$
ii) $|vxy| \leq p$
iii) $|vy| > 0$
iv) $\forall i = 0, 1, 2, \ldots \ uv^i xy^i z \in L$

To prove a language is not a CFL, one might show that there exists a string w such that no such parsing exists.

Beware: As is the case with regular languages, it is possible for a language to 'pass' the tandem pumping lemma but **not** be context-free.

1.2 PDA Definitions

(a) A *pushdown automaton* (PDA) is similar to a non-deterministic finite automaton, except for an additional stack. It is represented as a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$

- Q is the set of states,
- Σ is the input alphabet (and $\Sigma_\epsilon = \Sigma \cup \{\epsilon\}$),
- Γ is the stack alphabet (and $\Gamma_\epsilon = \Gamma \cup \{\epsilon\}$),
- $\delta : Q \times \Sigma_\epsilon \times \Gamma_\epsilon \to P(Q \times \Gamma_\epsilon)$ is the transition function

- $q_0 \in Q$ is the start state
- $F \subseteq Q$ is the set of accept states

(b) A pushdown automaton $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$ accepts input w if there exists an accepting computation of M on w. In more detail, M accepts w if w can be written as $w = w_1 w_1 w_2 \ldots w_m$, where each $w_i \in \Sigma_\epsilon$ and sequence of states $r_0, r_1, \ldots, r_m \in Q$ and strings $s_0, s_1, \ldots, s_m \in \Gamma^*$ exist that satisfy the following three conditions. The strings s_i represent the sequence of stack contents that M has on the accepting branch of the computation.

1. M starts out properly, in the start state and with and empty stack: $r_0 = q_0$ and $s_0 = \epsilon$.
2. M moves properly according to the state, stack, and next input symbol: For $i = 0, \ldots, m-1$, we have $(r_{i+1}, b) \in \delta(r_i, w_{i+1}, a)$, where $s_i = at$ and $s_{i+1} = bt$ for some $a, b \in \Gamma_\epsilon$ and $t \in \Gamma^*$
3. An accept state occurs at the end: $r_m \in F$

(c) A state diagram for a pushdown automaton can be drawn as shown below. The transition is taken as read "a" from input, pop "b" from stack, push "c" to stack

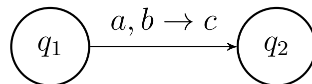


Figure 1: Example diagram for pushdown automaton

(d) **Fact:** Pushdown automata are equivalent in power to context-free grammars. A language L is context free if and only if there exists a pushdown automaton that recognizes L.

(e) Note that this immediately gives yet another proof that every regular language is context free. This is because for every regular language there exists a NFA that recognizes it, therefore there exists a PDA that recognizes it (the PDA can just ignore the stack / always pop ϵ and push ϵ to stack, and otherwise do the same as the NFA in terms of state transitions).

1.3 Closure properties

We saw in class some useful closure properties, corresponding to the regular operations:

(a) If L_1 and L_2 are context-free, so is $L_1 \cup L_2$

(b) If L_1 and L_2 are context-free, so is $L_1 \circ L_2$

(c) If L is context-free, so is L^*.

In contrast, some of the closure properties of regular languages do not work with context-free languages:

(a) If L_1 and L_2 are context-free, $L_1 \cap L_2$ might not be.

(b) If L is context-free, $\overline{L}$ (which we use to denote the complement of L) might not be.

Caution: with closure properties, we can only use what we prove. In particular, for context-free L_1 and L_2, $L_1 \cap L_2$ might be context free! But it also might not be.

As a simple example, consider a CFL L (over the alphabet Σ) as well as the regular language Σ^* (which is also a CFL since every regular language is a context free language). Consider, then, that $L \cap \Sigma^* = L$ is once again context-free.

Similarly, if any language L is regular, then $\overline{L}$ is also regular. But both languages are also context-free.

Exercise 1: As a more interesting example, show that the complement of $L = \{a^i b^i : i \geq 0\}$ is context-free.

Solution: How can a string w fail to be in L?

- One possibility is w is not of the form $a^* b^*$.
- Another is $w = a^i b^j$ with $i < j$.
- Finally, could have $w = a^i b^j$ with $i > j$.

So $\overline{L}$ is the union of these three cases. Since CFLs are closed under union, if we make a grammar that generates each case, we'll be done.

For the first case, we wish to generate the language $L_1 = \{w : w \notin a^*b^*\}$. This is the complement of a regular language, so must be context-free. The grammar below works:

$$\begin{aligned} T &\longrightarrow UbUaU \\ U &\longrightarrow \epsilon \mid aU \mid bU \end{aligned}$$

For the second case, we wish to generate $L_2 = \{a^ib^j : i < j\}$. We use a grammar similar to the one for $\{a^ib^i\}$, but add a rule that allows us to add extra b's at the end:

$$B \longrightarrow b \mid aBb \mid Bb$$

The third case for $L_3 = \{a^ib^j : i > j\}$ is very similar:

$$A \longrightarrow a \mid aAb \mid aA$$

So if we use the grammar from the union construction that combines these rules, we will generate the desired language $\overline{L}$:

$$\begin{aligned} S &\longrightarrow T \mid B \mid A \\ T &\longrightarrow UbUaU \\ U &\longrightarrow \epsilon \mid aU \mid bU \\ B &\longrightarrow b \mid aBb \mid Bb \\ A &\longrightarrow a \mid aAb \mid aA \end{aligned}$$

We claim that the grammar generates $\overline{L}$ as desired. As we noted at the beginning of the problem, $w \in \overline{L}$ if and only if $w \in L_1 \cup L_2 \cup L_3$, and we use the union construction, so if we justify that our grammars for L_1, L_2, and L_3 work, we are done.

For L_1, we note that U generates all strings over $\{a, b\}^*$. So the language generated by T corresponds to the language of the regular expression $(a \cup b)^* b (a \cup b)^* a (a \cup b)^*$. We can see that this is exactly the complement of the language generated by the regular expression $a^* b^*$ as desired.

For L_2, we see that at any point in a computation there is only a single non-terminal B, and we can only generate a's before this B and b's after the B. Since we also have the rule $B \to b$, the final step in a derivation for a string $w \in L_2$ will be of the form

$$a^* B b^* \Longrightarrow a^* b b^*$$

so that w is of the form $a^* b^*$. Further, since each rule either keeps the number of a's and b's equal, or adds one b and no a's, we see a string derived from the grammar must be of the form $a^i b^j$ with $i < j$.

On the other hand, for any string of the form $w = a^i b^j$ with $i < j$, we can write $w = a^n b^n b^k$ with $n \geq 0$ and $k > 0$. We see that if we use the first rule, $B \to aBb$, n times we will get

$$B \overset{*}{\Longrightarrow} a^n B b^n$$

then applying the second rule $B \to Bb$ $k - 1$ times, and the rule $B \to b$ once, we will get

$$B \overset{*}{\Longrightarrow} a^n B b^n \overset{*}{\Longrightarrow} a^n b^k b^n = w$$

The argument that we have the right grammar to generate L_3 is very similar. So we conclude our grammar indeed generates $\overline{L}$. Note: this is not a full formal proof (which would involve induction), but is a pretty thorough justification.

Exercise 2: A further nice closure property is that if L_1 is context-free and L_2 is *regular*, then $L_1 \cap L_2$ is context-free. Can you prove it? (Hint: use PDAs).

Solution: Since L_1 is context free, there's a PDA $M_1 = (Q_1, \Sigma, \Gamma, \delta_1, q_1, F_1)$ recognizing it. Since L_2 is regular, there's a NFA $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognizing it.

We construct a PDA for $L_1 \cap L_2$, by using a product construction for the finite state part and transitions, keeping track of both M_1 and M_2 states simultaneously, and using the stack in the same way that M_1 uses its stack

(note that M_2 does not use the stack). Thus, when in some state (r_1, r_2) (corresponding to r_1 in M_1 and r_2 in M_2), reading a on the input and with b on the stack: if M_1 would switch from r_1 to r'_1, and would have popped b and pushed c to stack, and if M_2 would switch from r_2 to r'_2, our new PDA would switch to (r'_1, r'_2), pop b and push c to stack.

Formally, the new PDA is $M = (Q_1 \times Q_2, \Sigma, \Gamma, \delta, (q_1, q_2), F_1 \times F_2)$, where δ is defined as follows.

$$\delta((r_1, r_2), a, b) = \bigcup_{(r'_1, c) \in \delta_1(r_1, a, b), r'_2 \in \delta_2(r_2, a)} \{((r'_1, r'_2), c)\}$$

With this construction, we can argue that there exists a computation of the PDA M on a string w that ends in state (r_1, r_2) if and only if there exists a computation of M_1 on w that ends in state r_1 and a computation of M_2 on w that ends in state r_2. Since the accepting states of M were defined to be $F_1 \times F_2$ (namely all the pairs (r_1, r_2) where $r_1 \in F_1, r_2 \in F_2$), it follows that a string w has an accepting computation by M if and only if it has accepting computations by both M_1 and M_2. Thus, M recognizes $L_1 \cap L_2$.

Note: can you see where the above proof would fail if we were trying to construct a PDA using two PDAs for two context free langauges?

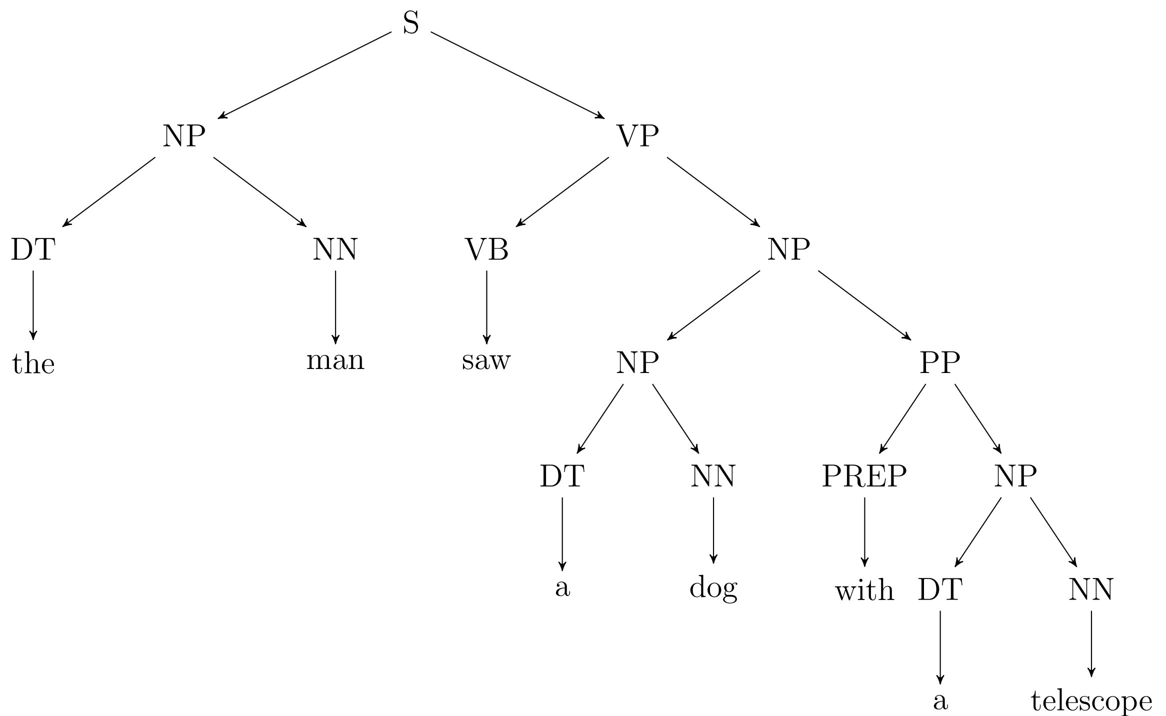
1.4 CFG Example

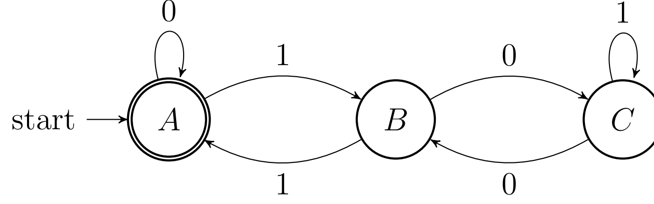
Consider a CFG with the following rules. This grammar generates a subset of English sentences. Here are what the variables stand for: NP = Noun Phrase, VP = Verb Phrase, DT = Determiner, PP = Prepositional Phrase, NN = Noun, VB = Verb, PREP = Preposition.

$$
\begin{aligned}
S &\longrightarrow NP\ VP \\
NP &\longrightarrow DT\ \ NN \mid NP\ \ PP \\
VP &\longrightarrow VB\ \ NP \mid VP\ \ PP \\
PP &\longrightarrow PREP\ \ NP \\
DT &\longrightarrow the \mid a \\
NN &\longrightarrow man \mid dog \mid telescope \\
VB &\longrightarrow saw \\
PREP &\longrightarrow with
\end{aligned}
$$

We will demonstrate the ambiguity derived from prepositional phrase attachment in English using the sentence *the man saw a dog with a telescope*.

Figure 2: PP attached to NP





The corresponding grammar from the construction described above would bet $G = (\{A, B, C\}, \{0, 1\}, R, A)$ with production rules R given by:

$$\begin{aligned} A &\longrightarrow 0A \mid 1B \mid \epsilon \\ B &\longrightarrow 0C \mid 1A \\ C &\longrightarrow 0B \mid 1C \end{aligned}$$

Justification: We claim that $L(G) = L(M)$. First, suppose $w \in L(G)$, where $w = w_1 w_2 \ldots w_n$ with each $w_i \in \Sigma$. If $w \in L(G)$, there is a computation such that $Q_0 \overset{*}{\Longrightarrow} w$. Inspecting all of the rules we added to R, we see there will always be at most a single non-terminal, and it will be the rightmost character of the current string. Further, we see the last production rule to generate w must be of the form $Q_i \to \epsilon$ for some $Q_i \in F$. We can only apply such a production once, at the last step of the derivation of w, because this rule will result in a string with no nonterminals. Building a string from the production rules will build from the beginning of w to the end, adding a single character to the prefix at each step. In view of all of these considerations, a derivation of w must look like:

$$Q_0 \Longrightarrow w_1 Q_{i_1} \Longrightarrow w_1 w_2 Q_{i_2} \overset{*}{\Longrightarrow} w_1 \ldots w_n Q_{i_n} \Longrightarrow w_1 \ldots w_n = w$$

where each derivation is of the form $Q_{i_k} \to w_{k+1} Q_{i_{k+1}}$. for $k = 0, \ldots, n-1$, and $Q_{i_n} \to \epsilon$ for $k = n$. But because of how we defined R, this means precisely that there is a computation on the DFA M given by

$$Q_0 \xrightarrow{w_1} Q_{i_1} \xrightarrow{w_2} Q_{i_2} \xrightarrow{w_3} \cdots \xrightarrow{w_n} Q_{i_n}$$

with $Q_{i_n} \in F$. This is an accepting computation of w on M, so we conclude $w \in L(M)$.

On the other hand, suppose $w \in L(M)$. Then there exists a computation

$$Q_0 \xrightarrow{w_1} Q_{i_1} \xrightarrow{w_2} Q_{i_2} \xrightarrow{w_3} \cdots \xrightarrow{w_n} Q_{i_n}$$

with Q_{i_n}. But this means precisely that we can apply the rule corresponding to each computation $\delta(Q_{i_k}, w_{k+1}) = Q_{i_{k+1}}$ to get a derivation in G:

$$Q_0 \Longrightarrow w_1 Q_{i_1} \Longrightarrow w_1 w_2 Q_{i_2} \stackrel{*}{\Longrightarrow} w_1 \dots w_n Q_{i_n} \Longrightarrow w_1 \dots w_n = w$$

So that $Q_0 \stackrel{*}{\Longrightarrow} w$, and $w \in L(G)$. We conclude $L(M) = L(G)$. Again, this argument is not a completely formal proof, but should convince you that our construction works.

We have shown that for every language L recognized by a DFA, there is a grammar that produces L. In fact, the CFG we constructed is right-linear. Thus, we've shown that any DFA can be transformed to an equivalent right-linear CFG. We conclude regular languages $\subset$ context-free languages.
Extra: The above construction shows that any DFA corresponds to a right-linear grammar. On the other hand, given a right-linear grammar, we can actually reverse the construction above to get an equivalent NFA.

2 Additional Problems

1. Consider the CFG $G_1 = (V, \Sigma, R, S)$ with $V = \{S\}$, $\Sigma = \{'(', ')'\}$, start variable S, and production rules R given by

$$S \to SS \,|\, (S) \,|\, \epsilon$$

   What language does this grammar generate?

2. Consider the CFG $G_2 = (V, \Sigma, R, S)$ with $V = \{S\}$, $\Sigma = \{a, b, c\}$, start variable S, and with production rules R given by

$$S \to aS \,|\, aSbS \,|\, c$$

   This grammar models **if-then** and **if-then-else** statements in programming languages where a stands for **if-condition-then**, b for **else**, and c for some other statement. Is the language generated by G_2 regular? Is G_2 ambiguous?

3. Consider the CFG $G_3 = (V, \Sigma, R, S)$ with $V = \{S, B\}$, $\Sigma = \{a, b\}$, with start variable S and production rules R given by

$$S \to aBa$$

$$B \to BB \,|\, b \,|\, \epsilon$$

What language does this grammar generate?

4. For the alphabet $\Sigma = \{a, b, c, d\}$, define the language

$$L = \{cw | w \in \{a, b\}^*, w = w^R\} \cup \{dw | w \in \{a, b\}^*\}$$

Prove that L is context free.

3 Solutions to Additional Problems

Problem 1

Consider the CFG $G_1 = (V, \Sigma, R, S)$ with $V = \{S\}$, $\Sigma = \{'(', ')'\}$, start variable S, and production rules R given by

$$S \to SS \,|\, (S) \,|\, \epsilon$$

What language does this grammar generate?

Solution: This grammar generates language L containing all strings with balanced parenthesis. We provide a full formal proof (not necessary on HW) that $L = L(G_1)$ using induction:

(a) To show $L(G) \subseteq L$, we do induction on number of productions.
Base Case: G_1 generates ϵ in 1 step.
Inductive Case: Let inductive hypothesis be that if $w \in \{(,)\}^*$ and G generates w in fewer than n derivations, then $w \in L$. We consider two cases for the first rule in the derivation:

- $S \to SS$. $S \Rightarrow SS \Rightarrow^* xS \Rightarrow^* xy$. Since both x and y are produced in fewer than n steps, they are in L by inductive hypothesis. Since the concatenation of two strings with balanced parentheses results in a string with balanced parentheses, we have that $w = xy \in L$.

- $S \to (S)$. $S \Rightarrow (S) \Rightarrow^* (x)$. Since x is produced in fewer than n steps, it is in L by inductive hypothesis. And $w = (x) \in L$ since it also has balanced parenthesis.

(b) To show $L \subseteq L(G)$, we induct on the length of $w \in L$.
Base Case: If $|w| = 0$, we have $w = \epsilon \in L$. Then by the rule $S \to \epsilon$, we have $w \in L(G)$.
Inductive Case: Let inductive hypothesis be that if $w \in L$ and $|w| < n$, then $w \in L(G)$. We consider a string with $|w| = n$ in 2 cases:

- $w = (x)$. By inductive hypothesis x is produced by G, so we can use $S \Rightarrow (S) \Rightarrow^* (x)$ to produce w.
- $w = xy$ where $x, y \in L$ and $|x| < n$ and $|y| < n$, then by inductive hypothesis x and y can both be produced by G, meaning x and y both have balanced parenthesis. So we can use $S \Rightarrow SS \Rightarrow^* xy$ to produce w.
- w must have the one of the two forms above since if w cannot be divided into $w = xy$ with $|x| < n$ and $|y| < n$, then the parentheses are not balanced in every proper prefix of w, which means in every proper prefix there are more (than) and w itself must start with (and end with).

We can now conclude that $L(G) = L$ since $L(G) \subseteq L$ and $L \subseteq L(G)$.

Problem 2

Consider the CFG $G_2 = (V, \Sigma, R, S)$ with $V = \{S\}$, $\Sigma = \{a, b, c\}$, start variable S, and with production rules R given by

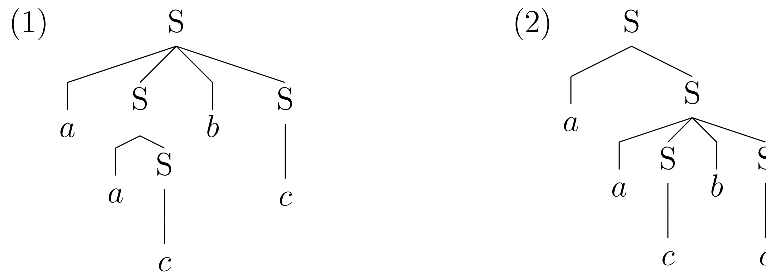
$$S \to aS \mid aSbS \mid c$$

This grammar models **if-then** and **if-then-else** statements in programming languages where a stands for **if-condition-then**, b for **else**, and c for some other statement. Is the language generated by G_2 regular? Is G_2 ambiguous?

Solution: We note from looking at the grammar that a string in $L(G_2)$ must have at most as many b's as a's in the string (this is because every production rule that adds a b also adds an a to the string).

$L(G_2)$ is **not regular**. Assume towards contradiction that it was. Let p be the pumping length for $L(G_2)$. Consider the string $w = a^p c(bc)^p$. $w \in L(G_2)$, since it can be produced by applying the rule $S \to aSbS$ to the leftmost S for p times, which results in $a^p S(bS)^p$, and then applying $S \to c$ for all the remaining S. We also have that $|w| = 3p + 1 \geq p$. Consider any decomposition of w into xyz satisfying the three criteria of the pumping lemma for regular languages. Since y is non-empty and xy is of length at most p, it must be that $y = a^m$ for some $0 < m \leq p$. We choose $i = 0$. But the string xz cannot be in L because it has more occurrences of b's than a's and every string in $L(G_2)$ has at most as many b's as a's. We have thus demonstrated a contradiction to our assumption that $L(G_2)$ is regular.

This grammar **is ambiguous** (which is similar to the "if-then-else" example given in class). Consider the string $aacbc$ in $L(G_2)$, which has two distinct leftmost derivations:
(1) $S \to aSbS \to aaSbS \to aacbS \to aacbc$
(2) $S \to aS \to aaSbS \to aacbS \to aacbc$
The grammar is therefore ambiguous. If we think about what kind of **if-then-else** statement this string represents, we see it corresponds to "if condition then if condition then statement else statement". The first derivation matches the "else" statement with the first "if", while the second derivation matches the "else" statement with the second "if". As parse trees:



Note: the language generated by this grammar is not inherently ambiguous. In other words, there is a non-ambiguous CFG that generates the same language. This is good, as it allows us to unambiguously parse if-then-else statements in actual programming languages!

Problem 3

Consider the CFG $G_3 = (V, \Sigma, R, S)$ with $V = \{S, B\}$, $\Sigma = \{a, b\}$, with start variable S and production rules R given by

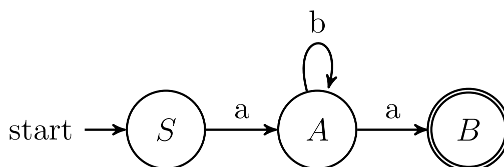
$$S \to aBa$$

$$B \to BB \,|\, b \,|\, \epsilon$$

What language does this grammar generate?

Solution: The grammar generates $L(ab^*a)$. To see this, we note that from the start variable we must use the production rule $S \to aBa$. Then, we see that B generates all strings of the form b^*: on one hand, the only terminal in any rule involving B is a b, so it's clear that the strings generated by B are a subset of $L(b^*)$. On the other hand, any string of this form is generated by B: to get b^n for $n > 0$ from B, we apply the first rule $B \to BB$, $n-1$ times to get a string B^n, and then we apply the rule $B \to b$ for each variable. We can also apply $B \to \epsilon$ initially to get ϵ, so the strings generated by B are exactly those generated by the regular expression b^*. We conclude $L(G_3) = L(ab^*a)$.

Extra: Building off of what we said about right-linear grammars in section 2, note that if we construct an NFA that recognizes $L(G_3) = L(ab^*a)$:



then we can easily follow the construction in the proof to create a right-linear grammar G_3' recognizing the same language. Let $G_3' = (V, \Sigma, R, S')$ with the same V and Σ, and grammar rules

$$\begin{aligned} S' &\longrightarrow aA \\ A &\longrightarrow bA \mid aB \\ B &\longrightarrow \epsilon \end{aligned}$$

Problem 4

For the alphabet $\Sigma = a, b, c, d$, define the language

$$L = \{cw | w \in \{a, b\}^*, w = w^R\} \cup \{dw | w \in \{a, b\}^*\}$$

Prove that L is context free.

Solution: We will construct a PDA for the language (another approach that also works is to construct a CFG).

The following PDA recognizes L. It is the same as the PDA constructed in class that accepts palindromes, with the addition of a transition function in the beginning to check if the first character in the input string is a "c" or a "d". If the first character is a "c," run the PDA that recognizes palindromes over the alphabet $\{a, b\}$. If the first character is a "d," accept any other sequence of a's and b's (if another c or d appear in the string, there is no transition so the computation "dies" and won't accept).

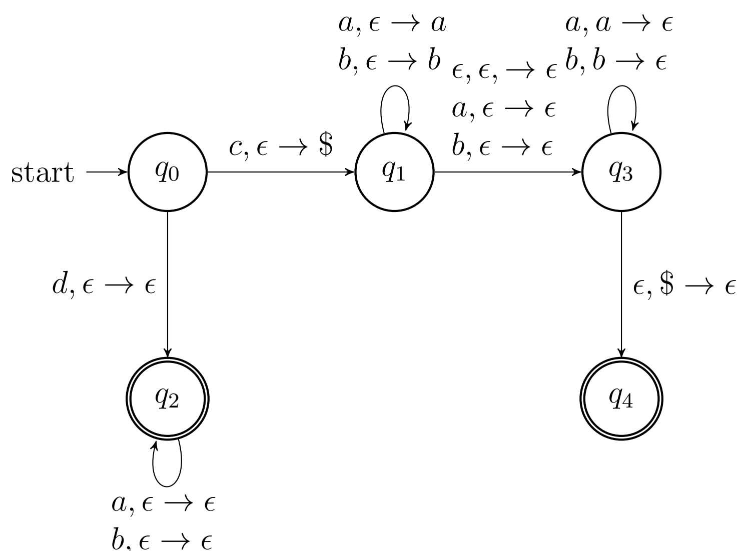


Figure 3: PDA that accepts L

4 Applications / Motivation

In this section, we will discuss the motivations behind understanding context-free languages, and the applications of these understandings. This is not

required material, but you might find it interesting.

4.1 What does "Context-Free" mean?

To start off, you might be wondering why this set of languages is called "context-free" ... what is the 'context' and what exactly makes these languages 'free' of it? To put it simply, 'context' refers to the symbols to the left and right of a non-terminal symbol as one is deriving a string. For example, if we have the CFG:

$$S \to aBa$$

$$B \to BB \,|\, b \,|\, \epsilon$$

So, let's say we are creating a string in the language of this CFG, and we currently have the string $abBa$. When looking at the non-terminal B, we don't care about the symbols to the left and right of B – they have no bearing on how we will replace B. This is what makes the language "context-free"... when we see a non-terminal, the symbols around it do not dictate the possible ways in which we can replace it.

What would it look like if we *did* care about the context of non-terminals? Well, that is what we call a *context-sensitive language (CSL)*. In similar fashion to context-free **grammars**, we can make *context-sensitive grammars (CSG)*. In a CSG, all rules are of the form:

$$\alpha A \beta \to \alpha \gamma \beta$$

where

$$A \in V$$

$$\alpha, \beta \in (V \cup \Sigma)^*$$

$$\gamma \in (V \cup \Sigma) \circ (V \cup \Sigma)^*$$

In words, what this rule would mean is that we can replace A with γ when we see the patterns (of non-terminals and terminals potentially) α to the **left** of A and β to the **right** of A. (Additionally, we may have a rule $S \to \varepsilon$ to allow production of ε, if S doesn't appear on the righthandside of other rules).

We will not cover CSG's in this course, but it turns out:

$$\{L \mid L \text{ is a } CFL\} \subsetneq \{L \mid L \text{ is a } CSL\}$$

4.2 What are some applications of CFL's / CFG's

CFL's are most prevalent in the fields of Linguistics and Natural Language Processing (NLP). CFG's are good models for many languages, such as English (consider the CFG from section 1.4). In this context, we can make our non-terminals generally correspond to grammatical groups of words (nouns, verbs) and our terminals correspond to specific words themselves. In NLP, *probabilistic context-free grammars (PCFG)* are often used to express the probability that a non-terminal follows a rule (for example, the probability that the **NOUN** non-terminal turns into *chicken*). In this case, one must make sure that the probabilities of all the rules originating from a non-terminal add up to 1.

Interestingly, however, there are some languages which have structures that cannot be captured by CFL's.

For example, in Dutch, it is possible to have this kind of structure, which loosely mirrors the non-context free language: $\{a^n b^m c^n d^m\}$:

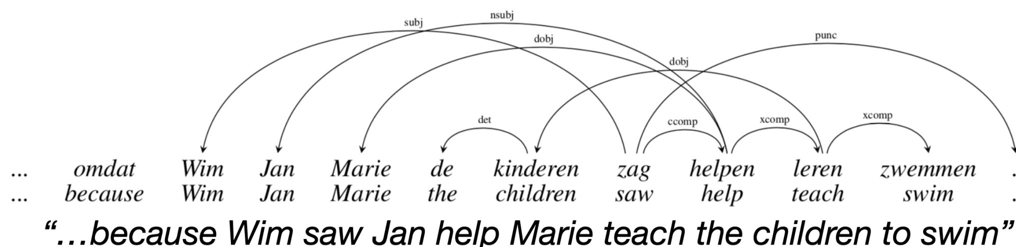


Figure 4: Credit – Professor Daniel Bauer, NLP Fall 2022 Slides