COMS W3261: Theoretical Computer Science.
Instructor: Tal Malkin

Problem Set 6

Due: Thur, 04/07/11.

1. (a) Construct a Turing machine which decides the language $L = \{w | w$ contains the string ab exactly once$\}$ over the alphabet $\Sigma = \{a, b\}$.

   Give a detailed formal description of the Turing Machine. The transition function of your TM can be described either with a table, or with a state diagram. To accompany the formal description, provide whatever high-level / English explanations are necessary to make sure that it is clear how your machine works.

   (b) Provide the sequence of configurations of your TM when started on each of the following four inputs: $abaab, aaba, \epsilon, bb$.

2. Consider the following high-level description of a Turing Machine T, which expects an input of the form [G], where [G] is some way to encode a directed graph as an input string.

   T: " On input [G]:

   1. Check that [G] is an encoding of a directed graph with at most one outgoing edge from each node.
   If it's not of this form, *reject*.

   2. Select the first node of G and mark it with two marks, corresponding to 'visited' and to 'current'.

   3. If the node u marked as 'current' has an outgoing edge to to another node v, move the 'current' mark from u to v. Mark v as 'visited' (if it's not already marked this way). Go to 3.

   4. If the node u marked as 'current' does not have any outgoing edge, scan the input for the first node v that is not marked as 'visited'. If there is such a v, move the 'current' mark from u to v, mark v as visited, and goto 3.
   If there is no such v, *accept*."

   (a) Convince yourself that with enough time and patience, you could provide an appropriate encoding [G] and a detailed formal description of the Turing Machine T described above. Then write "I'm convinced" (no need to give any details or justification.)

   Hints: To see another example regarding a graph problem, though not related to this one, check example 3.23 in the book. That example also suggests a way to encode an undirected graph. Convince yourself that you can also encode a directed graph G. In order to mark the nodes, you may add to the tape alphabet, in addition to the regular alphabet, three more versions of each symbol: one marked as 'visited', one marked as 'current' and one marked as both.

   (b) What is the language recognized by T? (That is, which inputs are accepted by T)?

(c) Is T a recognizer? Explain your answer. If your answer is no, then give a high-level description of a recognizer for the same language.

Is T a decider? Explain your answer. If your answer is no, then give a high-level description of a decider for the same language.

3. For this problem you can use high-level descriptions of any Turing Machine (algorithm) that you provide.

   (a) Prove that the class of Turing-recognizable languages is closed under the concatenation operation.

   (b) Prove that the class of Turing-decidable languages is closed under the complementation operation.

   (c) Does your proof of part (a) work to prove the same claim for the class of Turing-decidable languages? Does your proof of part (b) work to prove the same claim for the class of Turing-recognizable languages? Explain your answers.

4. Extra credit: Prove that a language C is TM-recognizable if and only if there exists a decidable language D such that $C = \{x \mid \exists y \text{ s.t. } (x, y) \in D\}$.