Fun with Mathematical Inequalities—Week 4

CSHP Fall 2021

Problem 1
Recall that for $x, y \in \mathbb{R}^n$ we defined

$$
\|x - y\|_p := \left( \sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p}.
$$

For $p = 0$, we defined

$$
\|x - y\|_0 := \# \{ i : x_i \neq y_i \}
$$

and for $p = \infty$, we defined

$$
\|x - y\|_{\infty} := \max_i |x_i - y_i|.
$$

We showed in class that $\|x - y\|_0$ is a metric (i.e. it satisfies non-negativity, symmetry, and the triangle inequality). Show that $\|x - y\|_{\infty}$ is also a metric.

Problem 2
Recall that we defined the unit $p$-ball $B_p$ as

$$
B_p := \{ x : \|x - 0\|_p = 1 \}.
$$

We drew what $B_p$ looked like for $p = 1$ and $p = 2$ in two-dimensions. What do $B_0$ and $B_{\infty}$ look like? What does $B_{3/2}$ look like?

Problem 3
Show that for $x, y, z \geq 0$, we have

$$
x + y + z \leq \frac{\sqrt{x^2 + y^2} + \sqrt{z^2 + y^2} + \sqrt{x^2 + z^2}}{\sqrt{2}}.
$$