Convex Influences

Shivam Nadimpalli
(Columbia)

Joint work with

Anindya De
(Penn)

Rocco Servedio
(Columbia)
An Intriguing Analogy

Monotone $f : \{0, 1\}^n \to \{0, 1\}$

(If $x \preceq y$ then $f(x) \leq f(y)$)

Symmetric Convex $K \subseteq \mathbb{R}^n$

(If $x \in K$, then $-x \in K$)

• Several algorithmic and structural parallels in recent years.
• This work: A new notion of influence for symmetric convex sets.
Monotone **Boolean Functions & Symmetric Convex Sets**

**Analytic Structure**
- Fourier concentration up to degree $O(\sqrt{n})$
- Fourier Hermite concentration up to degree $O(\sqrt{n})$

**Density Increment Results**
- **Kruskal–Katona**
  - A Kruskal–Katona Analogue

**Uniform Distribution Weak Learning**
- $\mathcal{N}(0, 1)^n$
  - $\Omega\left(\frac{\log n}{\sqrt{n}}\right)$ advantage for poly-time learners
  - $\Omega\left(\frac{1}{\sqrt{n}}\right)$ advantage for poly-time learners

**Uniform Distribution Strong Learning**
- $\mathcal{N}(0, 1)^n$
  - $n^{O(\sqrt{n})}$ time for constant accuracy
  - $n^{O(\sqrt{n})}$ time for constant accuracy

**Correlation Inequalities**
- **The Harris–Kleitman Inequality**
- Royen’s Gaussian Correlation Inequality
The Harris–Kleitman Inequality (1966)

Recall $f : \{0, 1\}^n \to \{0, 1\}$ is monotone if $x \preceq y \implies f(x) \leq f(y)$.

$x_i \leq y_i$ for all $i$

**Theorem:** For $f, g : \{0, 1\}^n \to \{0, 1\}$ monotone, we have

$$\mathbb{E}[f \cdot g] - \mathbb{E}[f] \cdot \mathbb{E}[g] \geq 0$$

where expectations are with respect to the uniform distribution on $\{0, 1\}^n$. 
Royen’s Inequality (2014)

- Also called the Gaussian Correlation Inequality (GCI).
- Open for over 40 years.

$K \subseteq \mathbb{R}^n$ is symmetric if $x \in K$ implies $-x \in K$.

**Theorem:** Let $K, L \subseteq \mathbb{R}^n$ be symmetric convex sets, identified with their 0/1 indicator functions. Then

$$\mathbb{E}[K \cdot L] - \mathbb{E}[K] \cdot \mathbb{E}[L] \geq 0$$

where the expectations are with respect to $\mathcal{N}(0, 1)^n$. 
Royen’s Inequality (2014)

Equivalently: If $\gamma_n(\cdot)$ is the $n$-dimensional, standard Gaussian measure, then for symmetric convex $K, L \subseteq \mathbb{R}^n$

$$\gamma_n(K \cap L) \geq \gamma_n(K) \cdot \gamma_n(L).$$
Towards Quantitative Inequalities

- Harris–Kleitman as well as Royen’s inequalities are qualitative statements:

\[
\mathbb{E}_D[f \cdot g] - \mathbb{E}_D[f] \cdot \mathbb{E}_D[g] \geq 0.
\]

- Can we hope to get a better lower bound?

Perhaps in terms of some property of \( f \) and \( g \)?
Definition: Given a Boolean function \( f : \{0, 1\}^n \rightarrow \{0, 1\} \), we define the influence of coordinate \( i \in [n] \) on \( f \) as

\[
\text{Inf}_i[f] := \Pr_{x \sim \{0, 1\}^n} \left[ f(x) \neq f(x \oplus i) \right]
\]

where \( x \oplus i := (x_1, \ldots, x_{i-1}, 1 - x_i, x_{i+1}, \ldots, x_n) \).
Towards Quantitative Inequalities

• Harris–Kleitman as well as Royen’s inequalities are qualitative statements:

\[
\mathbb{E}_D[f \cdot g] - \mathbb{E}_D[f] \cdot \mathbb{E}_D[g] \geq 0.
\]

• Can we hope to get a better lower bound? Perhaps in terms of some property of \( f \) and \( g \)?
Talagrand’s Correlation Inequality (1996)

**Theorem:** Let $f, g : \{0, 1\}^n \to \{0, 1\}$ be monotone. Then

$$\mathbb{E}[f \cdot g] - \mathbb{E}[f] \cdot \mathbb{E}[g] \geq \frac{1}{C} \cdot \Psi \left( \sum_{i=1}^{n} \text{Inf}_i[f] \cdot \text{Inf}_i[g] \right)$$

where $C$ is a universal constant and $\Psi(x) := \frac{x}{\log(e/x)}$.

- Several applications of Talagrand’s lemma in additive combinatorics, analysis of Boolean functions, etc.
**Story So Far**

<table>
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<th>Monotone $f, g : {0, 1}^n \rightarrow {0, 1}$</th>
<th>Symmetric Convex $K, L \subseteq \mathbb{R}^n$</th>
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<td>Uniform Distribution</td>
<td>$\mathcal{N}(0, 1)^n$</td>
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**H-K:** $\mathbb{E}[fg] - \mathbb{E}[f] \mathbb{E}[g] \geq 0$

**Talagrand:** $\mathbb{E}[f \cdot g] - \mathbb{E}[f] \cdot \mathbb{E}[g]$

\[
\geq \frac{1}{C} \cdot \Psi \left( \sum_{i=1}^{n} \text{Inf}_i[f] \cdot \text{Inf}_i[g] \right)
\]

where $\Psi(x) := \frac{x}{\log(e/x)}$

**Royen:** $\mathbb{E}[K \cdot L] - \mathbb{E}[K] \cdot \mathbb{E}[L] \geq 0$

**DNS '21:** $\mathbb{E}[K \cdot L] - \mathbb{E}[K] \cdot \mathbb{E}[L]$

\[
\geq \frac{1}{C} \cdot \Phi \left( \sum_{i=1}^{n} \tilde{K}(2e_i) \cdot \tilde{L}(2e_i) \right)
\]

where $\Phi(x) := \frac{x}{\log^2(1/x)}$
**Definition:** Given a symmetric convex set $K$ and $v \in S^{n-1}$, we define

$$\text{Inf}_v[K] := \mathbb{E}_{x \sim \mathcal{N}(0,1)^n} \left[ K(x) \cdot \left( 1 - \frac{\langle v, x \rangle^2}{\sqrt{2}} \right) \right]$$

$$= -h_2(\langle x, v \rangle)$$

to be the **convex influence** of $v$ on $K$. 

**Convex Influence**
Convex Influence in Terms of Dilations

**Proposition:** Take \( v = e_1 \) and write

\[
M_\delta := \text{diag}(1 + \delta, 1, \ldots, 1).
\]

Then we have

\[
\Inf_{e_1}[K] = \frac{1}{\sqrt{2}} \cdot \lim_{\delta \to 0} \frac{\gamma_n(M_\delta K) - \gamma_n(K)}{\delta}.
\]

- Analogous to the **Russo–Margulis Lemma** for Boolean functions.
- Informally, convex influence captures rate of growth under dilations.
Convex Influence in Terms of Dilations

\[ \text{Inf}_{e_1}[K] = \frac{1}{\sqrt{2}} \cdot \lim_{\delta \to 0} \frac{\gamma_n(M_\delta K) - \gamma_n(K)}{\delta} \]
**Total Convex Influence**

**Definition:** For symmetric convex $K$, write $K_{1+\delta} := K + \delta K$ and define

$$I[K] := \frac{1}{\sqrt{2}} \cdot \lim_{\delta \to 0} \frac{\gamma_n(K_{1+\delta}) - \gamma_n(K)}{\delta}$$

to be the total convex influence of $K$.

- Equivalently, we have $I[K] := \sum_{i=1}^{n} \text{Inf}_{e_i}[K]$ and so

$$I[K] = \mathbb{E}_{x \sim \mathcal{N}(0,1)^n} \left[ K(x) \cdot \left( \frac{n - \|x\|^2}{\sqrt{2}} \right) \right].$$

- Follows that $I[K]$ is rotationally invariant.
Total Convex Influence

\[
I[K] = \frac{1}{\sqrt{2}} \cdot \lim_{\delta \to 0} \frac{\gamma_n (\square)}{\delta}
\]
A Sanity Check

Is convex influence non-negative?

Boolean $\text{Inf}_i[f] \in [0, 1]$
Are Convex Influences Non-Negative?

\[ \text{Inf}_{e_1}[K] = \frac{1}{\sqrt{2}} \cdot \lim_{\delta \to 0} \frac{\gamma_n(M_\delta K) - \gamma_n(K)}{\delta} \]
Convex Influences Are Non-Negative

Fact: $\gamma_{n-1}(K_t)$ is log-concave in $t$.

In particular, it is symmetric and unimodal about 0 and so

$$\gamma_n(M_\delta K) = 2 \int_0^\infty \gamma_{n-1}(K_{(1-\delta)t}) d\gamma_1(t)$$

$$\geq 2 \int_0^\infty \gamma_{n-1}(K_t) d\gamma_1(t)$$

$$= \gamma_n(K).$$

Follows from dilation definition of influence that $\text{Inf}_{e_1}[K] \geq 0$. 

$K_t := \{x \in K : x_1 = t\}$
A Sanity Check

Is convex influence non-negative? Yes!

Boolean $\text{Inf}_i[f] \in [0, 1]$

What does zero convex influence mean?

If $\text{Inf}_i[f] = 0$ then $f(x)$ independent of $x_i$
Zero Convex Influences Implies Cylinder

Proposition: If $\text{Inf}_v[K] = 0$, then $K(x) = K(y)$ whenever $x_v \perp = y_v \perp$ almost surely.
A Sanity Check

Is convex influence non-negative? **Yes!**

Boolean $\mathbf{Inf}_i[f] \in [0, 1]$

What does zero convex influence mean? **Cylinder along direction!**

If $\mathbf{Inf}_i[f] = 0$ then $f(x)$ independent of $x_i$

Analogs of Boolean functions?

Dictatorships, Majority, Tribes?
Boolean Dictatorships are like Slabs

**Dict** 1 \( (x) := x_1 \)

If \( i \neq 1 \) then \( \text{Inf}_1[\text{Dict}_1] = 0 \)

\( K := \{ x \in \mathbb{R}^n : |x_1| \leq 1 \} \)

If \( i \neq 1 \) then \( \text{Inf}_{e_i}[K] = 0 \)
Majority is like the Ball

\[ \text{Maj}_n(x) := 1(\sum_{i=1}^{n} x_i \geq n/2) \]

\[ \text{Inf}_i[\text{Maj}_n] = \Theta\left(\frac{1}{\sqrt{n}}\right) \]

\[ \mathbb{B}^n := \{ x \in \mathbb{R}^n : \|x\|_2 \leq \sqrt{n} \} \]

\[ \text{Inf}_v[\mathbb{B}^n] = \Theta\left(\frac{1}{\sqrt{n}}\right) \]
Tribes is like the Solid Cube

$\text{Tribes}_n(x) = \bigvee_{i=1}^s \left( \bigwedge_{j=1}^w x_{i,j} \right)$

$\text{Inf}_{i}[\text{Tribes}_n] = \Theta\left( \frac{\log n}{n} \right)$

$C := \left\{ x \in \mathbb{R}^n : |x|_{\infty} \leq \Theta\left( \sqrt{\log n} \right) \right\}$

$\text{Inf}_{e_i}[C] = \Theta\left( \frac{\log n}{n} \right)$
A Sanity Check

Is convex influence non-negative?  
Yep!

Boolean $\text{Inf}_i[f] \in [0, 1]$

What does zero convex influence mean?  
Cylinder along direction!

If $\text{Inf}_i[f] = 0$ then $f(x)$ independent of $x_i$

Analogs of Boolean functions?  
Yep!

Dictatorships, Majority, Tribes?
Our Main Results

A Poincaré Inequality for Convex Influence

A KKL Analogue

Sharp Thresholds for Convex Sets à la Friedgut–Kalai

A Robust Kruskal–Katona for Convex Sets à la O’Donnell–Wimmer

A Weak Friedgut Junta Theorem
The KKL Theorem

**Theorem:** If $f : \{0, 1\}^n \rightarrow \{0, 1\}$ has $\inf_i f \leq \varepsilon$ for all $i$, then

$$I[f] = \Omega \left( \text{Var}[f] \cdot \log \left( \frac{1}{\varepsilon} \right) \right).$$

- Improvement of the Poincaré inequality and tight for the Tribes$_n$ function.
- Proof relies on hypercontractivity.
Theorem: If $K \subseteq \mathbb{R}^n$ is a symmetric convex set with $\inf_v [K] \leq \varepsilon$ for all $v \in S^{n-1}$ and $\varepsilon \leq \operatorname{Var}[K]$, then

$$I[K] = \Omega \left( \operatorname{Var}[K] \sqrt{\log \left( \frac{\operatorname{Var}[K]}{\varepsilon} \right)} \right).$$

- Quadratically weaker than Boolean KKL.
- Proof relates total convex influence to the inradius $r_{in}(K)$ of $K$. 
Proof Sketch

\[ I[K] = \frac{1}{\sqrt{2}} \lim_{\delta \to 0} \frac{\gamma_n(K) - \gamma_n((1 - \delta)K)}{\delta} \]

Gaussian Isoperimetry

\[ I[K] = \Omega(\text{Var}[K] \cdot r_{\text{in}}) \]

Brascamp–Lieb Inequality

\[ r_{\text{in}}(K) = \Omega\left(\sqrt{\log \left(\frac{\text{Var}[K]}{\varepsilon}\right)}\right) \]
Several Open Directions

A Stronger KKL for Convex Influences? Matched by the Solid Cube

Friedgut’s Junta Theorem for Convex Influences?

Are Low Influence Directions Almost Irrelevant?

Algorithmic Results? Convexity Testing?
Thanks for watching!