Is Your Unitary Low Dimensional?

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Informally, if unitary $U$ acts non-trivially on only $k$ of the $n$ qubits...

$$U = \begin{bmatrix}
  V & 0 & 0 & 0 \\
  0 & V & 0 & 0 \\
  0 & 0 & \ddots & 0 \\
  0 & 0 & 0 & V \\
\end{bmatrix}_{2^n \times 2^n}$$

**Definition:** Unitary $U$ acting on $n$ qubits is a $k$-junta if $U = V \otimes I$ for some $2^k \times 2^k$ unitary $V$. 
A Natural Question

\[ |\psi\rangle \xrightarrow{\mathcal{O}_U} U |\psi\rangle \]

**Question:** Given query access to a unitary \( U \) on \( n \) qubits and \( k > 0 \), is \( U \) a \( k \)-junta?

**Goal:** Minimize \# of queries to \( \mathcal{O}_U \)
A Natural Question

|ψ⟩ → \( \mathcal{O}_U \) → \( U |ψ⟩ \)

**Question:** Given query access to a unitary \( U \) on \( n \) qubits and \( k > 0 \), is \( U \) a \( k \)-junta?

- **Goal:** Minimize \# of queries to \( \mathcal{O}_U \)
- Intuitively feels hard to do without dependence on \( n \)...
Relaxing The Question

(Old) Question: Given query access to a unitary $U$ on $n$ qubits and $k > 0$, is $U$ a $k$-junta?

Relax

(New) Question: Given query access to a unitary $U$ on $n$ qubits and $k > 0$, decide with high probability

- if $U$ is a $k$-junta; or
- if $U$ is far from every $k$-junta? (Hilbert-Schmidt distance, up to phase)
Property Testing: It’s Like Eggs, Only Harder!

Old Question

New Question

$k$-juntas

$\varepsilon$
The Question We Consider

(New) Question: Given query access to a unitary $U$ on $n$ qubits and $k > 0$, decide with high probability

- if $U$ is a $k$-junta; or
- if $U$ is far from every $k$-junta?

Important Special Case: Boolean $f : \{0, 1\}^n \rightarrow \{0, 1\}$
Prior Work

Unitary $U$

$f : \{0, 1\}^n \rightarrow \{0, 1\}$

$O(k)$
[AS07]

$O(k)$
[W11]

This work: $\tilde{O}(\sqrt{k})$
$\Omega(\sqrt{k})$

$\tilde{O}(\sqrt{k})$
[ABRdW16]

$\Omega(\sqrt{k})$
[BKT17]
Prior Work

Unitary $U$

$f : \{0, 1\}^n \rightarrow \{0, 1\}$

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Unitary $U$

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$\tilde{O}(\sqrt{k})$
[ABRdW16]

$\Omega(\sqrt{k})$
[BKT17]

This work: $\tilde{O}(\sqrt{k})$
$\Omega(\sqrt{k})$
Our Main Result

**Theorem:** You can decide (w.h.p.) if a unitary is a $k$-junta or “far” from a $k$-junta with $\tilde{O}(\sqrt{k})$ queries. Furthermore, this is essentially tight: testing if a unitary is a $k$-junta requires $\Omega(\sqrt{k})$ queries.
The Upper Bound

Recall from HW 1 that any $2^n \times 2^n$ matrix can be written as

$$U = \sum_{x \in \{I,X,Y,Z\}^n} \hat{U}(x)\sigma_x.$$

**Key Quantity:** The influence of a qubit $j$ on $U$ is defined as

$$\text{Inf}_j[U] = \sum_{x : x_j \neq I} |\hat{U}(x)|^2.$$  

- Informally captures how “non-trivially” $U$ acts on the qubit $j$.
- Naturally extends to $\text{Inf}_S[U]$ for $S \subseteq U$. 
The Upper Bound

**Key Lemma:** Can determine with $O(1/\sqrt{\delta})$ queries to $U$ and $U^\dagger$ if $\text{Inf}_S[U] \geq \delta$. 

$= O(\sqrt{k} \log k \log k \varepsilon)$ queries for testing quantum $k$-juntas.
The Upper Bound

**Key Lemma:** Can determine with $O(1/\sqrt{\delta})$ queries to $U$ and $U^\dagger$ if $\text{Inf}_S[U] \geq \delta$.

+ 

**[ABRdW16]:** Given unknown $A \subseteq [n]$ and access to

$$\text{Intersects}_A(S) := \begin{cases} 
1 & A \cap S \neq \emptyset \\
0 & \text{otherwise}
\end{cases},$$

can decide if $|A| \leq k$ or $|A| \geq k + d$ with $O(\sqrt{1 + k/d})$ queries.
The Upper Bound

**Key Lemma:** Can determine with $O(1/\sqrt{\delta})$ queries to $U$ and $U^\dagger$ if $\text{Inf}_S[U] \geq \delta$.

+ 

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$$= O\left(\frac{\sqrt{k \log k \log k}}{\varepsilon}\right)$$ queries for testing quantum $k$-juntas
The Lower Bound

[BKT17]: Testing if $f : \{0, 1\}^n \rightarrow \{0, 1\}$ is a $k$-junta requires $\Omega(\sqrt{k})$ queries.

- Encode $f : \{0, 1\}^n \rightarrow \{0, 1\}$ as $U_f := \text{diag}(-1^{f(x)})$.
- Essentially a reduction from [BKT17], but needs new structural result for unitary $k$-juntas.
Other Contributions

Learning unitary $k$-juntas:

- Upper bound via tomography
- A lower bound via communication complexity
Many thanks to
- Rocco Servedio and Xi Chen for helpful discussions, and
- All of you for listening 😊