CS Theory (Spring '25)	Assigned: February 11, 2025
Homework 2	
Instructor: *Josh Alman*	Due: February 20, 2025 at noon

Collaboration on homework is allowed. However, you must write up solutions by yourself and understand everything that you hand in. You may also consult other reference materials, but you may not seek out answers from other sources or use AI tools. See the course webpage for more details.

Write the solution to each part of each problem on a separate page. Please do not write your name on any page of the submission; we are using anonymous grading in GradeScope.

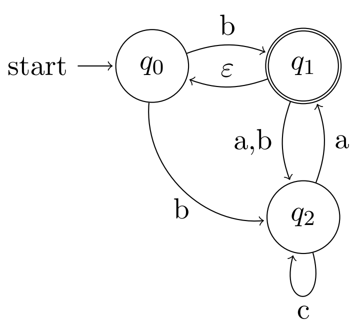
There are 100 total possible points, and also bonus questions worth an additional 20 points.

0 Exercises

Here are some recommended exercises from the Sipser textbook. You should not turn in your solutions for these, and we will not grade them. Exercises 1.7 (parts a,e,g,h), 1.11, 1.14b, 1.29 (parts a,b,c)

1 3-state NFAs (20 points; each part 10 points)

Let L_1 be the language recognized by the following NFA over the alphabet $\Sigma = \{\text{a,b,c}\}$:



(a) Give a DFA that recognizes L_1, and explain why it's correct.

(b) Consider the language $L_2 := \{1^x 0^y 1^z \mid x > 0, y \geq 0, z > 0\}$ over the alphabet $\Sigma = \{0, 1\}$. Give a 3-state NFA for L_2 and explain why it's correct.

2 Irregularities (20 points; each part 10 points)

Prove that each of the following languages over $\Sigma = \{0, 1\}$ is not regular.

(a) $\{0^n 1^m \mid n = 2m\}$.

(b) $\{0^n 1^m \mid n \neq 2m\}$.

3 Closure Properties (20 points; each part 10 points)

Throughout this question, let $\Sigma = \{a, b, c, d\}$. For a string $w \in \Sigma^*$ and a subset $S \subseteq \Sigma$, let $\text{only}_S(w)$ denote the string over S^* one gets by removing from w all characters not in S. For instance, if $w = abcdcacdb$, then $\text{only}_{\{a,b\}}(w) = abab$ and $\text{only}_{\{c,d\}}(w) = cdccd$ and $\text{only}_{\{a\}}(w) = aa$.

(a) Define the alphabets $\Sigma_1 = \{a, b\}$, $\Sigma_2 = \{c, d\}$. Prove that if L_1 is a regular language over Σ_1, and L_2 is a regular language over Σ_2, then the following is a regular language over Σ:

$$\{w \in \Sigma^* \mid \text{only}_{\{a,b\}}(w) \in L_1 \text{ and } \text{only}_{\{c,d\}}(w) \in L_2\}.$$

(b) Define the alphabet $\Sigma_3 = \{a, b, c\}$. Prove that if L is a regular language over Σ, then the following is a regular language over Σ_3:

$$\{w \in \Sigma_3^* \mid \text{ there is a } z \in L \text{ such that } \text{only}_{\{a,b,c\}}(z) = w\}.$$

(c) (10 BONUS points) Let $\Sigma_4 = \{0, 1\}$. For a string $w \in \Sigma_4^*$, let w^f denote the string over Σ_4^* gotten by changing every 0 to a 1 and every 1 to a 0. For instance, $110000101^f = 001111010$. Prove that if L is a regular language over Σ_4, then $\{w \in \Sigma_4^* \mid ww^f \in L\}$ is a regular language.

4 Pumpable Languages (10 points for part a)

For a language L over alphabet Σ, and a positive integer p, we say L is "*pumpable for length* p" if, for every $w \in L$ of length $\geq p$, there are strings x, y, z for which the following four conditions hold:

(1) $w = xyz$

(2) $|y| > 0$

(3) $|xy| \leq p$

(4) For all nonnegative integers i, we have $xy^iz \in L$.

In class, we proved that for every regular language L, there is a positive integer p such that L is pumpable for length p.

(a) Consider the language

$$L := \{w \in \{0, 1\}^* \mid \text{ the number of 1s in } w \text{ is a multiple of } 4\}.$$

Give, with proof, a specific value of p for which this language is pumpable for length p.

(b) (10 BONUS points) Suppose we changed condition (3) in the definition of "pumpable for length p" to just say $|y| \leq p$ instead of $|xy| \leq p$. Give (with proof!) an example of a language which is pumpable for length p for some positive integer p under this new definition, but is not pumpable for length p for any p under the original definition.

5 It's Finite for a Reason (10 points)

Recall that in a DFA, we require the set of states to be a finite set. Let's define a DIA (Deterministic Infinite Automaton) in the same way as a DFA, except that the set of states may be an infinite set. Prove that **every** language is recognized by a DIA. Please be sure to completely and unambiguously describe any DIAs that you construct! (Hint: similar to a problem on the previous homework, you may want to give a single DIA structure where just changing the accept states changes what the language is.)

6 New Power for DFAs (10 points)

Our goal when defining the NFA was to give a new, more powerful version of the DFA, but in some sense we failed: We proved in class that any language recognized by a NFA can also be recognized by a DFA. Give an example of a new, different way to augment a DFA with more power, along with an example of a language that is easier (in some sense) to compute using your new model. A high-level description and intuition is okay; no formal definition or proofs required, but please include an example picture of your new kind of DFA.

7 Collaborators, References, and Formatting (10 points)

Please list who you collaborated with on each problem, including any TAs or students you discussed the problems with in office hours. Also list any reference materials consulted other than the lectures and textbook for our class. Finally, please follow the formatting guidelines at the top of this document.