Lectures

- Lecture 12 (Thursday, July 25): Graph Neural Networks
- Lecture 13 (Monday, July 29): Adversarial Examples, Training Poisoning, DMI
- Lecture 14 (Tuesday, July 30): Reinforcement Learning
- Lecture 15 (Wednesday, July 31): Reinforcement Learning
- Lecture 16 (Thursday, August 1): Reinforcement Learning
- Lecture 17 (Monday, August 5): Deep Reinforcement Learning
- Lecture 18 (Tuesday, August 6): Deep Reinforcement Learning
- Lecture 19 (Wednesday, August 7): Imperfect Information Games
- Lecture 20 (Thursday, August 8): Automatic Deep Learning
- Lecture 21 (Monday, August 12): Deep Learning for Self Driving Cars
- Lecture 22 (Tuesday, August 13): Fair and Private Deep Learning
- Lecture 23 (Wednesday, August 14): Quantum Neural Networks
- Lecture 24 (Thursday, August 15): Project Presentations
Quantum Computing
Quantum Information Science

- Disruptive
- IBM research: access to IBM QX via the cloud
- Google 72-qubit quantum processor
- Microsoft Q#
Quantum Information Science

- Speculative

- Quantum supremacy in 2 years, advantage in 5 years, computers in 20 years.
Background

- Linear Algebra: linear combination of orthonormal basis, tensor product
- Probability
- Logic: gates and circuits
- Algorithms
- Geometry
Applications

- Cryptography
- Radar
- Computation
Spin

- Bit: 0/1
- Qubit: spin of electron, polarization of photon
Atom

- Positive nucleus
- Negative electrons orbiting
Atom

Atom as whole is magnet with S/N poles

Magnetic fields cancel each other

- Inner orbit: 2 electrons opposite directions
- 2nd orbit: 8 electrons
- 3rd orbit: 18 electrons
- 4th orbit: 18 electrons

Not cancelled

- 5th orbit: 1 electron for silver
Experiment: Stern and Gerlach

source of silver atoms

south magnet stronger

only two cases
Electron Spin

- Ask/measure direction: either yes in direction or in opposite direction
- Qubit -> measurement -> bit
- Repeat same measurement get same answer
- Measurements in different directions: measure vertical then horizontal 50/50
Electron Spin

- Repeat same measurements -> same answer
- Randomness occurs: sequence of measurements
- Measurements affect outcomes
  - Classical mechanics: throw ball, measure velocity, no effect of random photons on ball
  - Quantum mechanics: measuring electron spin effects spin, tiny particles.
Randomness

- Measure spin in vertical direction
- Then, measure spin in horizontal direction
- Random sequence of N,S,...
- Real randomness, no hidden variables
- Vs. coin flip: classical mechanics, sensitive dependence on initial conditions.
Polarization

- Polarized light experiment: more light through 3 than 2 sheets.
- Photons are polarized in two directions, orthogonal to direction of light travel.
Notation

- $|a\rangle$ column vector
- $\langle a|$ row vector

Linear algebra

- Are $|a_1\rangle,\ldots,|a_n\rangle$ an orthonormal basis? Iff $A^T A = I$
- Express $|y\rangle$ using orthonormal basis $|a_1\rangle,\ldots,|a_n\rangle$
  
  $|y\rangle = x_1|a_1\rangle + \ldots + x_n|a_n\rangle$ iff $\langle y| = x_1\langle a_1| + \ldots + x_n\langle a_n|$

$y = Ax$

$A^T y = A^T Ax$

$A^T y = x$

$\langle a_i|y\rangle = x_i$
What is the length of $|y>$?

$||y||^2 = <y|y> = (x_1|a_1> + ... + x_n|a_n>)(x_1|a_1> + ... + x_n|a_n>) = x_1^2 + ... + x_n^2$
Notation

Orthonormal basis:

- \( |\text{up}\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |\text{down}\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \)
- \( \langle\text{up}|\text{up}\rangle = 1, \quad \langle\text{down}|\text{down}\rangle = 1, \quad \langle\text{up}|\text{down}\rangle = 0, \quad \langle\text{down}|\text{up}\rangle = 0 \)

- \( |\text{right}\rangle = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, \quad |\text{left}\rangle = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \)
- \( \langle\text{right}|\text{right}\rangle = 1, \quad \langle\text{left}|\text{left}\rangle = 1, \quad \langle\text{right}|\text{left}\rangle = 0, \quad \langle\text{left}|\text{right}\rangle = 0 \)

- \( |\text{upright}\rangle = \begin{bmatrix} \frac{1}{2} \\ -\sqrt{3}/2 \end{bmatrix}, \quad |\text{downleft}\rangle = \begin{bmatrix} \sqrt{3}/2 \\ \frac{1}{2} \end{bmatrix} \)
- \( \langle\text{upright}|\text{upright}\rangle = 1, \quad \langle\text{downleft}|\text{downleft}\rangle = 1, \quad \langle\text{upright}|\text{downleft}\rangle = 0, \quad \langle\text{downleft}|\text{upright}\rangle = 0 \)
Spin State: Linear Combination

- $|y\rangle = x_0 |a_0\rangle + x_1 |a_1\rangle$ represented by orthonormal basis

Once measured:
- $|y\rangle$ jumps to $|a_0\rangle$ with probability $x_0^2$, $x_0 = <a_0|y>$
- $|y\rangle$ jump to $|a_1\rangle$ with probability $x_1^2$, $x_1 = <a_1|y>$
Electron Spin State

- $|y\rangle$
- Measure using orthonormal basis $|\text{up}\rangle$, $|\text{down}\rangle$
- Measurement results in $|\text{up}\rangle$
- Measure again using orthonormal basis $|\text{up}\rangle$, $|\text{down}\rangle$
- $|\text{up}\rangle = 1 |\text{up}\rangle + 0 |\text{down}\rangle$
- Measurement results in $|\text{up}\rangle$
- Measure using orthonormal basis $|\text{right}\rangle$, $|\text{left}\rangle$
- $|\text{up}\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = x_0 |\text{right}\rangle + x_1 |\text{left}\rangle = x_0 \begin{bmatrix} 1/sqrt(2) \\ -1/sqrt(2) \end{bmatrix} + x_1 \begin{bmatrix} 1/sqrt(2) \\ 1/sqrt(2) \end{bmatrix}$
- $|\text{xi}\rangle = \langle a_i | y \rangle$
  - $x_0 = \langle \text{up} | \text{right} \rangle = \begin{bmatrix} 1/sqrt(2), -1/sqrt(2) \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} = 1/sqrt(2)$
  - $x_0^2 = 1/2$
- $x_1 = \langle \text{up} | \text{left} \rangle = \begin{bmatrix} 1/sqrt(2), 1/sqrt(2) \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} = 1/sqrt(2)$
  - $x_1^2 = 1/2$
Linear Combination

- No measurement can distinguish between
  \[|\text{up}\rangle = x_0|a_0\rangle + x_1|a_1\rangle \text{ and } -|\text{up}\rangle = -x_0|a_0\rangle - x_1|a_1\rangle\]
- Or between \(|y\rangle\) and \(-|y\rangle\) since \(x_0^2 = (-x_0)^2\) and \(x_1^2 = (-x_1)^2\)
- so they are equivalent
Polarization Experiment

- Rotate polarized filter by angle beta
- Lets through photons polarized in direction beta
- Blocks photons polarized in direction orthogonal to beta
- Orthonormal basis $\begin{bmatrix} \cos(\beta) & \sin(\beta) \\ -\sin(\beta) & \cos(\beta) \end{bmatrix}$
Polarization Experiment

- 1st measurement: [1], [0] photons will be in state [1] [0] [1] [0]
- 2nd measurement: rotated 45 degrees [1/sqrt(2)] , [1/sqrt(2)] [-1/sqrt(2)] [1/sqrt(2)]
- \( [0] = \frac{1}{\sqrt{2}} [1/sqrt(2)] + \frac{1}{\sqrt{2}} [1/sqrt(2)] \)
- \( [1] = \frac{-1}{\sqrt{2}} [0] + \frac{1}{\sqrt{2}} [1/sqrt(2)] \)
- Probability of passing is ½, photons passing in state [1/sqrt(2)] [-1/sqrt(2)]
- 3rd measurement: [0], [1] [1] [0] [1/sqrt(2),-1/sqrt(2)] = -1/sqrt(2)[0] + 1/sqrt(2)[1]
- Probability of passing ½, lets through photons in state [1] [0]
Qubit

- $|y\rangle$ in $\mathbb{R}^2$
- Measurement introduces orthonormal basis $|a_0\rangle$, $|a_1\rangle$
- Qubit written as superposition of basis vectors same as vector written as linear combination of basis vectors
  
  $|y\rangle = x_0|a_0\rangle + x_1|a_1\rangle$

- After measurement qubit state jumps
  to $|a_0\rangle$ with probability $x_0$
  to $|a_1\rangle$ with probability $x_1$

- Associate $|a_0\rangle$ with 0 and $|a_1\rangle$ with 1

- Qubit has infinite possible values, once measured we get 0/1
Two Qubits

- Alice: orthonormal basis $|a_0>, |a_1>$
- Bob: orthonormal basis $|b_0>, |b_1>$
- Alice wants to send Bob a 0
- Alice sends qubit in state $|a_0>$
- Bob measures with respect to his basis
  \[ |a_0> = x_0|b_0> + x_1|b_1> \]
- State jumps to $|b_0>$ with probability $x_0^2$
- Jumps to $|b_1>$ with probability $x_1^2$
Secure Communication

- Alice and Bob want to communicate securely, Eve wants to eavesdrop.
- Alice sends Bob a stream of qubits
- Alice measures qubits using her orthonormal basis $|a_0>, |a_1>$
- Bob measures qubits that Alice sends him using his orthonormal basis $|b_0>, |b_1>$
- If Alice wants to send 0 she can send qubit in state $|a_0>$
- Bob measures with respect to his basis so $|a_0> = d_0|b_0> + d_1|b_1>$
- Qubit jumps
  - To $|b_0>$ with probability $d_0^2$ and Bob writes 0
  - To $|b_1>$ with probability $d_1^2$ and Bob writes 1
- If Alice and Bob use the same basis they will always receive same bit, however if Eve chooses same basis she will receive the message
- Therefore, Alice and Bob choose different basis
Secure Communication: Bennet & Brassard

- If Alice and Bob choose same bases then Bob will get same bit that Alice sent
- If Alice and Bob choose different bases then half the time Bob gets correct bit and half the time Bob gets wrong bit.
Secure Communication: BB84

- Alice chooses a key she wants to send to Bob
- 2 basis:
  - $V = [1,0]$ $H = [1/\sqrt{2}, 1/\sqrt{2}]$
  - $[0,1]$ $[-1/\sqrt{2}, 1/\sqrt{2}]$
- Key is string of bits used for encryption
- Alice chooses a basis $V$ or $H$ at random with equal probability
- Alice sends Bob qubit consisting of appropriate basis vector
  - If Alice wants to send Bob 0 and chooses $V$ she sends $[1,0]$.
    - If Alice wants to send Bob 0 and chooses $H$ she sends $[1/\sqrt{2}, -1/\sqrt{2}]$.
- If key string is length $4n$ binary bits then Alice stores string of $4n$ $V$s or $H$s.
Bob randomly chooses between basis $V$ and $H$ with equal probability
Bob measures qubit in chosen basis
Bob stores string of length $4n$ of bits: measurements
Bob stores string of length $4n$ of $V$'s, $H$'s: basis chosen
Alice and Bob choose basis at random
- Half the time they choose the same basis then Bob gets bit that Alice sent
- Half the time they choose different basis then
  - Half the time Bob gets right bit
  - Half the time Bob gets wrong bit
- Alice and Bob compare basis strings of $\sqrt{V}$s, $H$s over unencrypted line and keep bits where basis are same
  Erase bits where basis are different
- If Eve is not intercepting they get same string of bit of length $2^n$
BB84: Eve

- If Eve intercepts
- Consider only $2^n$ cases where Alice and Bob’s basis are the same
- Eve chooses basis at random
- Half the time $(n)$ Eve chooses the right basis which is the same for all three
- Half the time $(n)$ Eve chooses the wrong basis
  Eve sends Bob qubit
  Bob measures qubit and gets 0/1 with equal probability
  Bob gets the correct bit half the time $\frac{1}{2^n}$
Alice and Bob have strings of bits of length $2^n$.

If Eve is not intercepting the strings are identical and they use $n$ of the bits as key.

If Eve is intercepting she will choose wrong basis half the time, so a quarter of Bob’s bits will disagree with Alice and they know the line is insecure.
2 Qubits

- Alice has 1 qubit
  \[ |v> = x_0|a_0> + x_1|a_1> \]
- Bob has another qubit
  \[ |w> = y_0|b_0> + y_1|b_1> \]
- Tensor product
  \[ |v> \otimes |w> = x_0y_0|a_0>x|b_0> + x_0y_1|a_0>x|b_1> + x_1y_0|a_1>x|b_0> + x_1y_1|a_1>x|b_1> \]

\[ |vw> = r|00> + s|01> + t|10> + u|11> \]

\[ r^2 + s^2 + t^2 + u^2 = 1 \] probabilities

\[ ru = st = x_0y_0x_1y_1 \]
2 Qubits

- Represent \(|v\rangle\) and \(|w\rangle\) using the form

\[ r|a_0\rangle|b_0\rangle + s|a_0\rangle|b_1\rangle + t|a_1\rangle|b_0\rangle + u|a_1\rangle|b_1\rangle \]

- Allow any values of \(r,s,t,u\) such that \(r^2 + s^2 + t^2 + u^2 = 1\)

- If \(ru = st\) Alice and Bob’s qubits are not entangled

- If \(ru \neq st\) Alice and Bob’s qubits are entangled
Unentangled Qubits

- $|v\rangle|w\rangle = \frac{1}{2}\sqrt{2}|a_0\rangle|b_0\rangle + \frac{\sqrt{3}}{2}\sqrt{2}|a_0\rangle|b_1\rangle + \frac{1}{2}\sqrt{2}|a_1\rangle|b_0\rangle + \frac{\sqrt{3}}{2}\sqrt{2}|a_1\rangle|b_1\rangle$

- $ru = \frac{\sqrt{3}}{8} = st \rightarrow$ qubits are not entangled

If Alice and Bob both make measurements:

- 00 with probability $\frac{1}{8}$
- 01 with probability $\frac{3}{8}$
- 10 with probability $\frac{1}{8}$
- 11 with probability $\frac{3}{8}$
Unentangled Qubits

- $|v⟩⟨w⟩ = \frac{1}{2}\sqrt{2}|a₀⟩⟨b₀⟩ + \frac{\sqrt{3}}{2}\sqrt{2}|a₀⟩⟨b₁⟩ + \frac{1}{2}\sqrt{2}|a₁⟩⟨b₀⟩ + \sqrt{3}\frac{1}{2}\sqrt{2}|a₁⟩⟨b₁⟩ =
  
  |a₀⟩(|b₀⟩ + \frac{\sqrt{3}}{2}\sqrt{2}|a₀⟩⟨b₁⟩ + \frac{1}{2}\sqrt{2}|a₁⟩⟨b₀⟩ + \sqrt{3}\frac{1}{2}\sqrt{2}|a₁⟩⟨b₁⟩⟩ =
  
  (\frac{1}{\sqrt{2}}|a₀⟩ + \frac{1}{\sqrt{2}}|a₁⟩)(\frac{1}{2}|b₀⟩ + \sqrt{3}\frac{1}{2}|b₁⟩)

- If Alice measures first obtains 0/1 with probability $\frac{1}{2}$
- If Bob measures first obtains 0/1 with probability $\frac{1}{4}, \frac{3}{4}$
- Alice’s measurements have no effect on Bob’s measurements
- Bob’s measurements have no effect on Alice’s measurements
Entangled Qubits

- $|w> = 1/2|a_0> |b_0> + 1/2|a_0> |b_1> + 1/\sqrt{2}|a_1> |b_0> + 0|a_1> |b_1>$

- $ru = 0 \neq st = 1/2\sqrt{2} \rightarrow$ qubits are entangled

- If Alice and Bob both make measurements
  - $00$ with probability $1/4$
  - $01$ with probability $1/4$
  - $10$ with probability $1/2$
  - $11$ with probability $0$
Entangled Qubits

- \( |v\rangle |w\rangle = \frac{1}{2} |a_0\rangle |b_0\rangle + \frac{1}{2} |a_0\rangle |b_1\rangle + \frac{1}{\sqrt{2}} |a_1\rangle |b_0\rangle + 0 |a_1\rangle |b_1\rangle = |a_0\rangle (\frac{1}{2} |b_0\rangle + \frac{1}{2} |b_1\rangle) + |a_1\rangle (\frac{1}{\sqrt{2}} |b_0\rangle + 0 |b_1\rangle) = \frac{1}{\sqrt{2}} |a_0\rangle (\frac{1}{\sqrt{2}} |b_0\rangle + \frac{1}{\sqrt{2}} |b_1\rangle) + \frac{1}{\sqrt{2}} |a_1\rangle (|b_0\rangle + 0 |b_1\rangle) \)

- Terms in parentheses are different, qubits are entangled

- If Alice makes a measurement, will get 0/1 with probability \( \frac{1}{2} \)

- When Alice gets 0 her qubit jumps to \( |a_0\rangle \) and system jumps to \( |a_0\rangle (\frac{1}{\sqrt{2}} |b_0\rangle + \frac{1}{\sqrt{2}} |b_1\rangle) \) and Bob’s qubit becomes \( (\frac{1}{\sqrt{2}} |b_0\rangle + \frac{1}{\sqrt{2}} |b_1\rangle) \) and is no longer entangled with Alice’s

- Alice gets 1 her qubit jumps to \( |a_1\rangle \) system jumps to \( |a_1\rangle (|b_0\rangle + 0 |b_1\rangle) \) and Bob’s qubit becomes \( |b_0\rangle \) and is no longer entangled with Alice’s
# Classical Gates

<table>
<thead>
<tr>
<th>NOT</th>
<th>AND</th>
<th>OR</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P$</td>
<td>$Q$</td>
<td>$P \lor Q$</td>
</tr>
<tr>
<td>$T$</td>
<td>$T$</td>
<td>$T$</td>
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<tr>
<td>$F$</td>
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</tbody>
</table>

$\neg (\neg P \land \neg Q) = P \lor Q$
Classical Gates

\[
\begin{array}{ccc}
\text{XOR} & \text{NAND} \\
\begin{array}{ccc}
P & Q & P \oplus Q \\
T & T & F \\
T & F & T \\
F & T & T \\
F & F & F \\
\end{array} & \\
\begin{array}{ccc}
P & Q & P \Downarrow Q \\
T & T & F \\
T & F & T \\
F & T & T \\
F & F & T \\
\end{array}
\end{array}
\]

\[
(P \land \neg Q) \lor (\neg P \land Q) = P \oplus Q
\]

\[
\neg (\neg (P \land Q) \land (\neg (\neg P \land Q))) = P \oplus Q
\]
We can generate any Boolean function from (NOT,AND)

We can generate any Boolean function from NAND

\[
\neg P = P \uparrow P \quad \text{and} \quad P \land Q = (P \uparrow Q) \uparrow (P \uparrow Q)
\]
Quantum Gates

- Gate rotates qubit
- Measuring device is fixed
- Use natural basis
CNOT

\[
\begin{array}{c|c|c|c}
\text{input} & \text{output} \\
\hline
x & y & x & x \oplus y \\
\hline
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 \\
\end{array}
\]
- 2 CNOT gates in series reverse each other

\[ \times \Theta \times \Theta y = y \]
Extend CNOT to Qubits

\[
\begin{array}{c|c|c|c}
\text{input} & \text{output} \\
\hline
x & y & x \oplus y \\
\hline
|0\rangle & |0\rangle & |0\rangle & |0\rangle \\
|0\rangle & |1\rangle & |0\rangle & |1\rangle \\
|1\rangle & |0\rangle & |1\rangle & |1\rangle \\
|1\rangle & |1\rangle & |1\rangle & |0\rangle \\
\end{array}
\]
Extend CNOT to Qubits

\[
\begin{array}{c|c}
\text{input} & \text{output} \\
\hline
|00\rangle & |00\rangle \\
|01\rangle & |01\rangle \\
|10\rangle & |11\rangle \\
|11\rangle & |10\rangle \\
\end{array}
\]
\[
\text{CNOT} (r|00\rangle + s|01\rangle + t|10\rangle + u|11\rangle)
= r|00\rangle + s|01\rangle + u|10\rangle + t|11\rangle
\]
CNOT

input $|x0\rangle$ superposition

$\frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle$

$|0\rangle$

$\frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle$

CNOT permutes basis vectors, corresponds to orthogonal matrix

$CNOT(|00\rangle) = |00\rangle \quad CNOT(|11\rangle) = |11\rangle$

$CNOT(|1x0\rangle) \neq |xx\rangle$ for $x \neq |0\rangle, |1\rangle$

output $\{ \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle \}$

entangled state
Quantum Gates for 1 Qubit

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$I(a_0|0\rangle + a_1|1\rangle) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = a_0|0\rangle + a_1|1\rangle$

$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$Z(a_0|0\rangle + a_1|1\rangle) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = a_0|0\rangle - a_1|1\rangle$

$Z(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle) = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$Y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Hadamard Gate

\[ H = \left[ \begin{array}{cc} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{array} \right] = \frac{1}{\sqrt{2}} \left[ \begin{array}{c} 1 \\ -1 \end{array} \right] \]

\[ H |0\rangle = \frac{1}{\sqrt{2}} \left[ \begin{array}{c} 1 \\ -1 \end{array} \right] |0\rangle = \frac{1}{\sqrt{2}} |1\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \]

\[ H |1\rangle = \frac{1}{\sqrt{2}} \left[ \begin{array}{c} 1 \\ -1 \end{array} \right] |1\rangle = \frac{1}{\sqrt{2}} |1\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \]
IBM Q

Circuit composer

Gates

Operations

Subroutines

+ Add

q[0] |0⟩
q[1] |0⟩
q[2] |0⟩
q[3] |0⟩
q[4] |0⟩
c5
No Cloning Theorem

A gate $G$ cloning a qubit cannot exist if $G$ exists.

\[ |x\rangle \quad \text{and} \quad |\bar{x}\rangle \]

then \[ G(|00\rangle) = |10\rangle \]
\[ G(|10\rangle) = |11\rangle \]

\[ G\left(\frac{1}{\sqrt{2}}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle\right) = \left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\right) \]
\[ G\left(\frac{1}{\sqrt{2}}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle\right) = \frac{1}{\sqrt{2}}\left(1|00\rangle + |10\rangle + |11\rangle + |10\rangle + |11\rangle\right) \]

$G$ is linear since it is a matrix operation.

\[ G\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle\right) = \frac{1}{\sqrt{2}}G(|00\rangle) + \frac{1}{\sqrt{2}}G(|10\rangle) \]
\[ = \frac{1}{\sqrt{2}}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle \]

however

\[ \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \neq \frac{1}{2}(1|00\rangle + |10\rangle + |10\rangle + |11\rangle) \]

contradiction $\Box$
Bell Circuit

\[ |0\rangle \rightarrow H \rightarrow \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle \]

\[ B(|00\rangle) = \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle \]

\[ H(|0\rangle) = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \quad H(|1\rangle) = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \]

System of 2 qubits has state

\[ \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) |0\rangle = \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |10\rangle \]

CNOT flips |10\rangle to |11\rangle and |11\rangle to |10\rangle

\[ B(|10\rangle) = \frac{1}{\sqrt{2}} |10\rangle + \frac{1}{\sqrt{2}} |10\rangle \]
\[ B(|11\rangle) = \frac{1}{\sqrt{2}} |11\rangle - \frac{1}{\sqrt{2}} |11\rangle \]
\[ B(|11\rangle) = \frac{1}{\sqrt{2}} |11\rangle - \frac{1}{\sqrt{2}} |10\rangle \]

4 outputs are entangled and form an orthonormal basis

Bell basis
Circuit composer

Gates

<table>
<thead>
<tr>
<th>Gates</th>
<th>Operations</th>
<th>Subroutines</th>
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</thead>
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<tr>
<td>$H$</td>
<td>$</td>
<td>0\rangle$</td>
</tr>
<tr>
<td>$\dagger$</td>
<td>$\text{IF}$</td>
<td>$\text{z}$</td>
</tr>
<tr>
<td>$\text{cH}$</td>
<td>$\text{cY}$</td>
<td>$\text{cZ}$</td>
</tr>
<tr>
<td>$\text{cRz}$</td>
<td>$\text{cU1}$</td>
<td>$\text{cU3}$</td>
</tr>
</tbody>
</table>

IBM Q
Reverse Bell Circuit

- Transforms Bells basis back to standard basis
Superdense Coding

Two electrons with entangled spin state
\[ \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle \]
1 for Alice, 1 for Bob
Alice wants to send Bob 2 bits: 00 / 01 / 10 / 11
Alice sends Bob her electron problem:
Bob can get 1 bit from qubit sent from Alice since once Bob measures qubit he gets 0 / 1 solution:
Bob extracts 1 bit from electron sent by Alice and 1 bit from entangled electron he holds.
Superdense Coding

Bob needs to perform some measurement in all cases.

Alice passes her entangled electron through a gate.

\[
\frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle \quad \text{for } 00
\]

\[
\frac{1}{\sqrt{2}} |10\rangle + \frac{1}{\sqrt{2}} |01\rangle \quad \text{for } 01
\]

\[
\frac{1}{\sqrt{2}} |10\rangle - \frac{1}{\sqrt{2}} |01\rangle \quad \text{for } 10
\]

\[
\frac{1}{\sqrt{2}} |10\rangle - \frac{1}{\sqrt{2}} |01\rangle \quad \text{for } 11
\]

Bob passes electron sent from Alice and his electron through a reverse Bell circuit and gets 00/01/10/11.
Quantum Teleportation

Alice and Bob each have a electron in entangled state
\[ \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \]

Alice has another electron in state \( a|0\rangle + b|1\rangle \)

Alice and Bob want to change Bob’s electron so that it has state \( a|0\rangle + b|1\rangle \)

Alice sends Bob 2 bits (only even thought there are definitely many possibilities for her electron)

Alice sends her 2 qubits through a CNOT gate then applies Hadamard gate to top qubit, i.e. passing her 2 electrons through inverse Bell
Quantum Teleportation

Initial state describing 3 qubits:

\( \left( a |0\rangle + b |1\rangle \right) \otimes \left( \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle \right) \)

\( \frac{a}{\sqrt{2}} |000\rangle + \frac{b}{\sqrt{2}} |011\rangle + \frac{a}{\sqrt{2}} |100\rangle + \frac{b}{\sqrt{2}} |111\rangle \)

Alice applies CNOT gate to first 2 qubits:

\( \frac{a}{\sqrt{2}} |00\rangle \otimes (\frac{1}{\sqrt{2}} |01\rangle + \frac{b}{\sqrt{2}} |11\rangle) + \frac{b}{\sqrt{2}} |01\rangle \otimes (\frac{1}{\sqrt{2}} |10\rangle + \frac{a}{\sqrt{2}} |00\rangle) \)

\( \frac{a}{\sqrt{2}} |00\rangle \otimes (\frac{1}{\sqrt{2}} |01\rangle + \frac{b}{\sqrt{2}} |11\rangle) + \frac{b}{\sqrt{2}} |01\rangle \otimes (\frac{1}{\sqrt{2}} |10\rangle + \frac{a}{\sqrt{2}} |00\rangle) \)

Alice applies Hadamard to first qubit, changing |0\rangle to \( \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \) and |1\rangle to \( \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle \)

\( \frac{a}{\sqrt{2}} |0\rangle \otimes (\frac{1}{\sqrt{2}} |00\rangle + \frac{b}{\sqrt{2}} |01\rangle + \frac{a}{\sqrt{2}} |10\rangle + \frac{b}{\sqrt{2}} |11\rangle) \)

\( + \frac{b}{\sqrt{2}} |10\rangle \otimes (\frac{1}{\sqrt{2}} |00\rangle + \frac{a}{\sqrt{2}} |01\rangle + \frac{b}{\sqrt{2}} |10\rangle + \frac{a}{\sqrt{2}} |11\rangle) \)

\( + \frac{a}{\sqrt{2}} |11\rangle \otimes (\frac{1}{\sqrt{2}} |00\rangle + \frac{b}{\sqrt{2}} |01\rangle + \frac{a}{\sqrt{2}} |10\rangle + \frac{b}{\sqrt{2}} |11\rangle) \)

\( + \frac{b}{\sqrt{2}} |00\rangle \otimes (a |0\rangle + b |1\rangle) + \frac{a}{\sqrt{2}} |01\rangle \otimes (a |0\rangle + b |1\rangle) \)

\( + \frac{b}{\sqrt{2}} |10\rangle \otimes (a |0\rangle - b |1\rangle) + \frac{a}{\sqrt{2}} |11\rangle \otimes (a |1\rangle - b |0\rangle) \)
Quantum Teleportation

Alice measures her 2 electrons in standard basis getting \( |0\rangle/|10\rangle/|10\rangle/|11\rangle \) each with prob. \( \frac{1}{2} \).

- If Alice gets \( |0\rangle \) Bob's qubit jumps to \( |a0\rangle+b1\rangle \)
- If Alice gets \( |01\rangle \) Bob's qubit jumps to \( |a1\rangle+b0\rangle \)
- If Alice gets \( |10\rangle \) Bob's qubit jumps to \( |a0\rangle-b1\rangle \)
- If Alice gets \( |11\rangle \) Bob's qubit jumps to \( |a1\rangle-b0\rangle \)

Alice sends Bob 2 bits corresponding to her measurements to let Bob know which state he is in.

If Bob receives 00 he is in the correct state and does nothing.

If Bob receives 01 he applies gate X to his qubit.

If Bob receives 10 he applies gate Y to his qubit.

If Bob receives 11 he applies gate Z to his qubit.

Bob's qubit ends in state \( |a0\rangle+b1\rangle \), the state of the qubit that Alice wanted to teleport.

There is only 1 qubit in state \( |a0\rangle+b1\rangle \) during the process. Initially Alice has it, then Bob has it. Sending the 2 bits corresponding prevents interference transmission.
Superdense Coding and Quantum Teleportation

Superdense coding: Alice sends Bob 1 qubit to convey 2 bits.

Alice encodes using Pauli transformations.
Bob decodes using reverse Bell circuit.

Quantum teleportation: Alice sends Bob 2 bits to teleport 1 qubit.

Alice encodes using reverse Bell circuit.
Bob decodes using Pauli transformations.

China has teleported a qubit from Earth to space.

Quantum teleportation provides a way of transporting qubit from one place to another without transporting the particle that represents the qubit.
Error Correction

repetition code
repeat symbol 3 times
if Alice wants to send 0 she sends 000
if “ ” “ 1 ” ” 111
Bob decodes 000,001,010,100 as 000
“ ” 110,110,101,011 as 111
Bob receives b0b1b2
computes b0 ⊕ b1 and b0 ⊕ b2
if b0 = b1 ⊖ b2 then b0 ⊕ b1 = 0 and b0 ⊕ b2 = 1
if b0 = b2 ⊖ b1 then b0 ⊕ b1 = 1 and b0 ⊕ b2 = 0
if b0 ⊖ b1 = b2 then b0 ⊕ b1 = 1 and b0 ⊕ b2 = 1
if Bob gets
00 correct
01 flips b2
10 flips b1
11 flips b0
parity test tells us
Where the error is
not what the error is
Quantum Bit-Flip Correction

Alice wants to send the qubit $a|0\rangle + b|1\rangle$ to Bob.

Error: $a|0\rangle + b|1\rangle$ gets changed to $a|1\rangle + b|0\rangle$.

Alice cannot send 3 copies of qubit to Bob: no cloning.

Alice replaces $|0\rangle$ with $|000\rangle$ and $|1\rangle$ with $|111\rangle$.

Using 2 CNOT gates:

3 entangled qubits:

$$a|000\rangle + b|111\rangle$$

starts with state $(a|0\rangle + b|1\rangle)|0\rangle|0\rangle = a|000\rangle + b|100\rangle$.

1st CNOT gate changes state to $a|000\rangle + b|110\rangle$.

2nd CNOT gate changes state to $a|000\rangle + b|111\rangle$.

Alice sends 3 qubits to Bob who receives:

Correct $a|000\rangle + b|111\rangle$.

or Incorrect $a|100\rangle + b|011\rangle$ with error in 1st qubit.

Incorrect $a|010\rangle + b|101\rangle$ “ 2nd “

Incorrect $a|001\rangle + b|110\rangle$ “ 3rd “
Quantum Bit-Flip Correction

If Bob measures the 3 entangled qubits, the state becomes unentangled and Bob gets 3 qubits that are a combination of 0's and 1's, knowing the values of a and b.

Bob can detect the error and correct it without measuring the 3 entangled qubits.

Bob uses parity checks.

Bob receives $a|000⟩ + b|111⟩$.

Input for first 4 qubits is $a|000⟩ + b|111⟩$.

2 CNOT gates on 4th wire perform parity check.

If $a ⊗ b = 0$, 4th qubit will be in state $a|000⟩ + b|111⟩$.

If $a ⊗ b = 1$, 4th qubit will be in state $a|100⟩ + b|011⟩$.

4th qubit not entangled with step 3.
Quantum Bit-Flip Correction

Similarly for $5$th qubit: not entangled with others

- if $C_0 \oplus C_2 = d_0 \oplus d_2 = 0$ then $5$th qubit is $|0\rangle$
- if $C_0 \oplus C_1 = d_0 \oplus d_1 = 0$ then $5$th qubit is $|1\rangle$

Qubits 4 and 5 are not entangled with qubits 1, 2, 3

Bob measures qubits 4 and 5

- if he gets $00$ then correct
- if he gets $01$ then flips 3rd qubit by $X$ gate on 3rd
- if he gets $10$ then flips 2nd qubit by $X$ gate on 2nd
- if he gets $11$ then flips 1st qubit by $X$ gate on 6th wire

Error corrected and qubits are in state Alice sent them,
Quantum Neural Networks
Tensor Networks

Quantum Entanglement in Deep Learning Architectures

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Modern deep learning has enabled unprecedented achievements in various domains. Nonetheless, employment of machine learning for wave function representations is focused on more traditional architectures such as restricted Boltzmann machines (RBMs) and fully-connected neural networks. In this letter, we establish that contemporary deep learning architectures, in the form of deep convolutional and recurrent networks, can efficiently represent highly entangled quantum systems. By constructing Tensor Network equivalents of these architectures, we identify an inherent re-use of information in the network operation as a key trait which distinguishes them from standard Tensor Network based representations, and which enhances their entanglement capacity. Our results show that such architectures can support volume-law entanglement scaling, polynomially more efficiently than presently employed RBMs. Thus, beyond a quantification of the entanglement capacity of leading deep learning architectures, our analysis formally motivates a shift of trending neural-network based wave function representations closer to the state-of-the-art in machine learning.
Tensor Networks

\[ f^{\text{CNN}}_{\theta_l}(x_1, \ldots, x_K) = \sigma \left( W^{(l,1)} x_1 + \ldots + W^{(l,K)} x_K \right) \]

\[ f^{\text{AC}}_{\theta_l}(x_1, \ldots, x_K) = (W^{(l,1)} x_1) \odot \ldots \odot (W^{(l,K)} x_K) \]

\[ \theta_l = (W^{(l,1)}, \ldots, W^{(l,K)}) \]

(a)

FIG. 1. Convolutional networks: In each layer $l \in \{1, \ldots, L\}$ of a depth-$L$ convolutional network, convolution kernels of size $K$ are slid across the input maps, computing the layer outputs after every stride of $S$ steps. (a) CACs and common CNNs share the same architectural description, and the type of convolutional network is determined by the function which computes the layers’ outputs. The $\sigma(\cdot)$ function defining $f^{\text{CNN}}_{\theta_l}$ is some element-wise non-linearity in the form of a sigmoid or ReLU [26], and the operation denoted $\odot$ defining $f^{\text{AC}}_{\theta_l}$ stands for element-wise multiplication between vectors. The function parameters are the convolution weights matrices $\theta_l = (W^{(l,1)}, \ldots, W^{(l,K)})$. (b,c) The calculation of a non-overlapping convolutional network (for which $K = S$) assumes a tree structure. A deep CAC with a $K = S$ restriction is equivalent to a Tree TN with internal order-$(K + 1)$ tensors, obeying index-wise: $(a')_{i,j_1,\ldots,j_K} = W^{(l,1)}_{i,j_1} \cdots W^{(l,K)}_{i,j_K}$. (d,e) In overlapping-convolutional networks, which achieve state-of-the-art performance, the convolution kernel stride is $S = 1$ (the output is calculated for every step of the kernel) and kernels are of general size, with typical values of $K = 3$ or $K = 5$. These values result in overlap of convolution kernels along the computation. The 0-padding at the edges ensures that a layer’s output is produced also for the edge activations. The TN corresponding to the calculation of the overlapping CAC must account for the inherent reuse of information due to the overlaps. Therefore, it involves duplication of external TN indices, such that re-used data is generated again and again by the TN, which cannot simply copy-paste information. Thus, the portrayed recursive TN structure is received. The presentation is in ID form for clarity, and extensions to 2D are straightforward.
FIG. 2. Recurrent networks: In a recurrent network, an incoming input is integrated with an existing hidden state that the network computes from previous inputs. (a) RACs and common RNNs share the above architectural description, and the type of recurrent network is determined by the integration function. The function parameters are the hidden and input weights matrices $\theta_l = (W^{(H,l)}, W^{(I,l)})$, and an output weights matrix $W^O$. (b,c) The shallow RAC is equivalent to an MPS TN with internal order-3 tensors obeying index-wise:

$$a_{ijk} = W^{(H,1)}_{ij} W^{(I,1)}_{ik}.$$ $h_0^1$ is some initial hidden state. (d,e) A depth 2 RAC is represented by a ‘recursive MPS’ TN structure, which makes use of input duplication to circumvent an inherent inability of TNs to model information re-use. The TN in (e) is portrayed for the example of $N = 3$. 
Quantum Convolutional Neural Networks

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We introduce and analyze a novel quantum machine learning model motivated by convolutional neural networks. Our quantum convolutional neural network (QCNN) makes use of only $O(\log(N))$ variational parameters for input sizes of $N$ qubits, allowing for its efficient training and implementation on realistic, near-term quantum devices. The QCNN architecture combines the multi-scale entanglement renormalization ansatz and quantum error correction. We explicitly illustrate its potential with two examples. First, QCNN is used to accurately recognize quantum states associated with 1D symmetry-protected topological phases. We numerically demonstrate that a QCNN trained on a small set of exactly solvable points can reproduce the phase diagram over the entire parameter regime and also provide an exact, analytical QCNN solution. As a second application, we utilize QCNNs to devise a quantum error correction scheme optimized for a given error model. We provide a generic framework to simultaneously optimize both encoding and decoding procedures and find that the resultant scheme significantly outperforms known quantum codes of comparable complexity. Finally, potential experimental realization and generalizations of QCNNs are discussed.
Quantum CNN

Figure 1: (a) Simplified illustration of CNNs. A sequence of image processing layers—convolution (C), pooling (P), and fully connected (FC)—transforms an input image into a series of feature maps (blue rectangles), and finally into an output probability distribution (purple bars). (b) QCNNs inherit a similar layered structure. (c) QCNN and MERA share the same circuit structure, but run in reverse directions.
Towards Quantum Machine Learning with Tensor Networks

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(Dated: August 1, 2018)

Machine learning is a promising application of quantum computing, but challenges remain as near-term devices will have a limited number of physical qubits and high error rates. Motivated by the usefulness of tensor networks for machine learning in the classical context, we propose quantum computing approaches to both discriminative and generative learning, with circuits based on tree and matrix product state tensor networks that could have benefits for near-term devices. The result is a unified framework where classical and quantum computing can benefit from the same theoretical and algorithmic developments, and the same model can be trained classically then transferred to the quantum setting for additional optimization. Tensor network circuits can also provide qubit-efficient schemes where, depending on the architecture, the number of physical qubits required scales only logarithmically with, or independently of the input or output data sizes. We demonstrate our proposals with numerical experiments, training a discriminative model to perform handwriting recognition using a optimization procedure that could be carried out on quantum hardware, and testing the noise resilience of the trained model.
FIG. 1. The quantum state of $N$ qubits corresponding to a tree tensor network (left) can be realized as a quantum circuit acting on $N$ qubits (right). The circuit is read from top to bottom, with the yellow bars representing unitary gates. The bond dimension $D$ connecting two nodes of the tensor network is determined by number of qubits $V$ connecting two sequential unitaries in the circuit, with $D = 2^V$. 
FIG. 2. Discriminative tree tensor network model architecture, showing an example in which $V = 2$ qubits connect different subtrees. Figure (a) shows the model implementation as a quantum circuit. Circles indicate inputs prepared in a product state as in Eq. 1; hash marks indicate qubits that remain unobserved past a certain point in the circuit. A particular pre-determined qubit is sampled (square symbol) and its distribution serves as the output of the model. Figure (b) shows the tensor network diagram for the reduced density matrix of the output qubit.
FIG. 4. Discriminative tensor network model for the case of a matrix product state (MPS) architecture with $V = 2$ qubits connecting each subtree. The symbols have the same meaning as in Fig. [2]. An MPS can be viewed as a maximally unbalanced tree.
FIG. 5. Generative tree tensor network model architecture, showing a case with $V = 2$ qubits connecting each subtree. To sample from the model, qubits are prepared in a reference computational basis state $|0\rangle$ (left-hand side of circuit). Then $2V$ qubits are entangled via unitary operations at each layer of the tree as shown. The qubits are measured at the points in the circuit labeled by square symbols (right-hand side of circuit), and the results of these measurements provides the output of the model. While all qubits could be entangled before being measured, we discuss in Section IV the possibility performing opportunistic measurements to reduce the physical qubit overhead.
FIG. 6. Generative tensor network model for the case of a matrix product state (MPS) architecture with $V = 2$ qubits connecting each unitary. The symbols have the same meaning as in Fig. 5.
FIG. 7. Model architecture used in the experiments of Section III which is a special case of the model of Fig. 2 with one virtual qubit connecting each subtree. For illustration purposes we show a model with 16 inputs and 4 layers above, whereas the actual model used in the experiments had 64 inputs and 6 layers.
FIG. 10. Qubit-efficient scheme for evaluating (a) discriminative and (b) generative tree models with $V = 2$ virtual qubits and $N = 16$ inputs or outputs. Note that the two patterns are the reverse of each other. In (a) qubits indicated with hash marks are measured and the measurement results discarded. These qubits are then reset and prepared with additional input states. In (b) measured qubits are recorded and reset to a reference state $\langle 0 \rangle$. 
FIG. 11. Qubit-efficient scheme for evaluating (a) discriminative and (b) generative matrix product state models for an arbitrary number of inputs or outputs. The figure shows the case of $V = 3$ qubits connecting each node of the network. When evaluating the discriminative model, one of the qubits is measured after each unitary is applied and the result discarded; the qubit is then prepared with the next input component. To implement the generative model, one of the qubits is measured after each unitary operation and the result recorded. The qubit is then reset to the state $\langle 0 \rangle$. 
FIG. 12. Mapping of the generative matrix product state (MPS) quantum circuit with $V = 3$ to a bond dimension $D = 2^3$ MPS tensor network diagram. First (a) interpret the circuit diagram as a tensor diagram by interpreting reference states $\langle 0 \rangle$ as vectors $[1, 0]$; qubit lines as dimension 2 tensor indices; and measurements as setting indices to fixed values. Then (b) contract the reference states into the unitary tensors and (c) redraw the tensors in a linear chain. Finally, (d) merge three $D = 2$ indices into a single $D = 8$ dimensional index on each bond.
Thank you