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Genus distribution of graph amalgamations: self-pasting at root-vertices

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Abstract

Counting the number of imbeddings in various surfaces of each of the graphs in an interesting family is an ongoing topic in topological graph theory. Our special focus here is on a family of closed chains of copies of a given graph. We derive *double-root partials* for *open chains of copies* of a given graph, and we then apply a self-amalgamation theorem, thereby obtaining genus distributions for a sequence of *closed chains of copies* of that graph. We use *recombinant strands of face-boundary walks*, and we further develop the use of multiple *production rules* in deriving partitioned genus distributions.

1 Introduction

Counting imbeddings of a graph in various surfaces follows several paradigms. Recent contributions include some closed formulas, some recursions that are useful in the efficient construction of tables, and some lower bounds. Our focus here is on deriving recursions for the genus distribution of a family of closed chains of copies of a graph.

The ***self-amalgamation*** of a double-rooted graph (G, u, v) is the graph obtained from G by merging vertices u and v. We sometimes use a subscripted asterisk to denote the self-amalgamation operation:

$$*_{uv}(G, u, v) \;=\; (X, w)$$

where w is the merged vertex. ($*_{uv}$ is a mnemonic for "paste u to v".)

Example 1.1 Let $(\ddot{K}_4, u, v)$ be the graph formed from the complete graph K_4 by inserting midpoints u and v on two edges with no endpoints in common, with u and v as the co-roots. Figure 1.1 illustrates that the result of the self-amalgamation $*_{uv}(\ddot{K}_4, u, v)$ is isomorphic to the 4-wheel W_4.

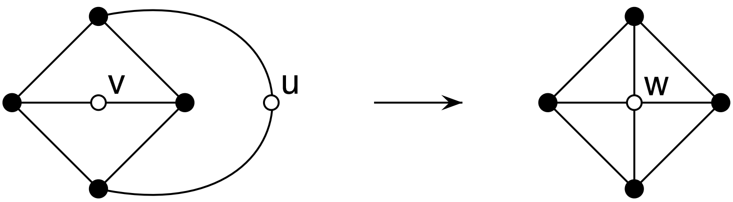

Figure 1.1: Self-amalgamation of $(\ddot{K}_4, u, v)$ yields W_4.

In the general case, where vertices u and v have arbitrary degrees, every imbedding of (X, w) ***induces*** a unique imbedding of (G, u, v), by which we mean the imbedding of G whose rotations at u and v are consistent with the rotation of vertex w in X and whose other rotations are the same as in X. We observe, conversely, that every imbedding of G ***induces*** a set of

$$deg(u) \cdot \binom{deg(v) + deg(u) - 1}{deg(u)} \tag{1.1}$$

imbeddings of $*_{uv}(G, u, v)$ (for details, see the proof of Proposition 1.1 of [13]), which we say are collectively the result of ***self-amalgamation of the imbedding***.

Scope of the Method

We observe that any Eulerian graph can be obtained from a set of cycle graphs by a sequence of amalgamations and self-amalgamations at 2-valent vertices. This suggests using the methods described here to count imbeddings of interesting families of Eulerian graphs. Moreover, since 2-valent vertices are obtainable by subdividing an edge, the results of this paper are also applicable to various families of non-Eulerian graphs.

Prior work on counting imbeddings of a graph in a minimum-genus surface includes [2], [8], [9], and [20]. Prior work on counting imbeddings in all orientable surfaces or in all surfaces includes [4], [5], [7], [10], [11], [12], [13], [14], [19], [21], [22], [23], [24], [26], [27], [28], [29], [30], [31], [32], [33], [34], and [35]. Complementary work on counting maps on a given surface is given by [6], [16], [17], [18], and many others.

This paper continues from [13] the usage of ***recombinant strands*** and ***production rules***. The abbreviation ***fb-walk*** means "face-boundary walk". Informally, we say "pasting" to mean any kind of amalgamation. In general the usage follows [15] and [1]. (Alternative background sources are [3], [25], and [36].)

2 Production Rules for Self-Amalgamations

A calculation of the genus distribution for the self-amalgamation of a graph has various similarities to the calculation for the amalgamation of two distinct graphs.

For instance, as per Formula (1.1), when a graph (G, u, v) has two 2-valent co-roots, the self-amalgamation $*_{uv}(G, u, v)$ has six imbeddings for each imbedding of (G, u, v); thus, the sum of the coefficients in the output of each production rule is 6, just as with pasting 2-valent roots from distinct graphs.

A dissimilarity arises from the possibility that one or both fb-walks at one co-root of a double-rooted graph may be incident on the other co-root, which could not happen with roots of two distinct graphs. The possibility of the two co-roots lying on the same fb-walk motivates us to introduce four new subpartials.

$$ds_i^0(G, u, v) = \text{the number of imbeddings of type-}ds_i\text{ such that neither fb-walk at }u\text{ is incident on }v;$$

$$ds_i'(G, u, v) = \text{the number of imbeddings of type-}ds_i\text{ such that one fb-walk at }u\text{ is (twice) incident on }v;$$

$$sd_i^0(G, u, v) = \text{the number of imbeddings of type-}sd_i\text{ such that the fb-walk at }u\text{ is not incident on }v;$$

$$sd_i'(G, u, v) = \text{the number of imbeddings of type-}sd_i\text{ such that the fb-walk at }u\text{ is (once) incident on }v;$$

Analysis of imbeddings of a self-amalgamations where no fb-walk is incident on both co-roots strongly resembles the analysis for amalgamating two distinct graphs, as indicated by Proposition 2.1.

Proposition 2.1 *Let (G, u, v) be a double-rooted graph with 2-valent co-roots. Then the following production rules hold for its self-amalgamation:*

$$dd_i^0 \longrightarrow 4g_{i+1} + 2g_{i+2} \tag{2.1}$$
$$ds_i^0 \longrightarrow 6g_{i+1} \tag{2.2}$$
$$sd_i^0 \longrightarrow 6g_{i+1} \tag{2.3}$$
$$ss_i^0 \longrightarrow 6g_{i+1} \tag{2.4}$$

Proof In a self-amalgamation of an imbedded graph $(G, u, v) \to S_i$, the number of vertices decreases by 1, while the number of edges remains the same. It follows from a simple Euler-characteristic calculation that, if the number of faces of a resultant imbedding increases by 1, then the resultant imbedding surface is S_i. Moreover, a decrease of 1 face implies an increase in genus to $i + 1$, and a decrease of 3 faces implies an increase in genus to $i + 2$.

Terminology In the absence of standard names for the various graphics we have selected to represent fb-walks in the figures, we assign names of colors to them, and we provide a legend.

Validations of production rules for imbeddings of amalgamations are by face-tracing (see [15]) on rotation projections. The amalgamation at the left of Figure 2.1 represents the four cases in which the two edge-ends at each co-root remain contiguous in the imbedded amalgamation. The strand we call *red* arises from an fb-walk at co-root u, and the strand we regard as *blue* arises from an fb-walk at co-root v. They are recombined, yielding a decrease of 1 in the number of faces. The amalgamation at the right represents the two cases where the edge-ends at u alternate with the edge-ends at v. Then all four strands recombine into a single fb-walk. This confirms the validity of Rule (2.1).

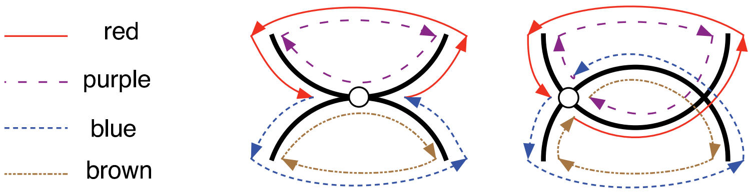

Figure 2.1: $dd_i^0 \longrightarrow 4g_{i+1} + 2g_{i+2}$

Similarly, Figure 2.2 illustrates the validation of Rule (2.2). Two fb-walks recombine into one in all four cases depicted at the left. A single red strand and a single blue strand recombine, and the number of faces decreases by 1.

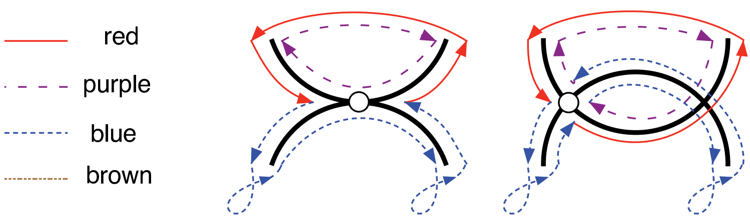

Figure 2.2: $ds_i^0 \longrightarrow 6g_{i+1}$

On the right, the blue fb-walk has been severed into two strands, one of which recombines with the single red strand, the other with the single so-called *purple* strand. Here, too, the number of faces decreases by 1. Reflecting these drawings thru a horizontal axis yields drawings that validate Rule (2.3).

Lastly, Figure 2.3 illustrates the validation of Rule (2.4). When edge-ends of each root are contiguous, a single red strand recombines with a single blue strand. When edge-ends of the respective co-roots alternate, the red fb-walk and the blue fb-walk both break into two strands. Face-tracing indicates that all four strands recombine into a single fb-walk. ♢

The complete set of Production Rules for self-amalgamations of a double-rooted graph with two 2-valent co-roots is given in Table 2.1. Theorem 2.2 establishes the correctness of the table.

Theorem 2.2 *All the production rules in Table 2.1 for a self-amalgamation* $*_{uv}(G, u, v) = (X, w)$ *are correct.*

Proof Proposition 2.1 establishes Rules (2.1), (2.2), (2.3), and (2.4). For the remaining rules, we describe some critical features of the drawings and leave them to the reader to draw.

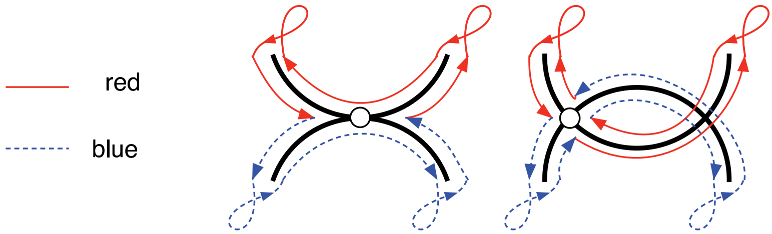

Figure 2.3: $ss_i^0 \longrightarrow 6g_{i+1}$

Table 2.1: The rules for a self-amalgamation $*_{uv}(G, u, v)$.

	production rule	reference
$dd_i^0 \longrightarrow$	$4g_{i+1} + 2g_{i+2}$	(2.1)
$ds_i^0 \longrightarrow$	$6g_{i+1}$	(2.2)
$sd_i^0 \longrightarrow$	$6g_{i+1}$	(2.3)
$ss_i^0 \longrightarrow$	$6g_{i+1}$	(2.4)
$dd_i' \longrightarrow$	$g_i + 5g_{i+1}$	(2.5)
$dd_i'' \longrightarrow$	$4g_i + 2g_{i+1}$	(2.6)
$ds_i' \longrightarrow$	$3g_i + 3g_{i+1}$	(2.7)
$sd_i' \longrightarrow$	$3g_i + 3g_{i+1}$	(2.8)
$ss_i^1 \longrightarrow$	$6g_i$	(2.9)
$ss_i^2 \longrightarrow$	$g_{i-1} + 5g_i$	(2.10)

There are three fb-walks altogether at the co-roots u and v when the rule

$$dd_i' \longrightarrow g_i + 5g_{i+1} \tag{2.5}$$

is invoked. They form four fb-walks in one rotation of the merged vertex w and two fb-walks in the five other rotations at w. There are only two fb-walks at u and v when the rule

$$dd_i'' \longrightarrow 4g_i + 2g_{i+1} \tag{2.6}$$

is applied. In four rotations at the merged vertex w there are three incident fb-walks, and only one fb-walk in two other rotations at w.

There are two fb-walks prior to amalgamation when either one of the rules

$$ds_i' \longrightarrow 3g_i + 3g_{i+1} \tag{2.7}$$

$$sd_i' \longrightarrow 3g_i + 3g_{i+1} \tag{2.8}$$

is applied. In three rotations the recombination of strands leads to three fb-walks at w, and at the other three rotations, they recombine into a single walk.

In cases ss_i^1 and ss_i^2, the operative rules are

$$ss_i^1 \longrightarrow 6g_i \tag{2.9}$$
$$ss_i^2 \longrightarrow g_{i-1} + 5g_i \tag{2.10}$$

and the same fb-walk has two incidences at vertex u and two at vertex v. When one of the strands at u contains both incidences on v (case ss_i^1), all six rotations cause recombinations of the strands into two fb-walks at merged vertex w. When both strands at u are incident on v (case ss_i^2), there is one rotation in which the respective edge-ends of u and v alternate, with all four strands recombining into a single fb-walk; in the other five rotations at w, there are two fb-walks. ♢

Corollary 2.3 *Let $(X, w) = *_{uv}(G, u, v)$ be a self-amalgamation, where roots u and v are both 2-valent. Then the genus distribution of graph X is given by the following equation (which omits writing (G, u, v) on the right):*

$$\begin{aligned} g_k(X) &= ss_{k+1}^2 \\ &+ dd_k' + 4dd_k'' + 3ds_k' + 3sd_k' + 6ss_k^1 + 5ss_k^2 \\ &+ 4dd_{k-1}^0 + 6ds_{k-1}^0 + 6sd_{k-1}^0 + 6ss_{k-1}^0 \\ &\quad +5dd_{k-1}' + 2dd_{k-1}'' + 3ds_{k-1}' + 3sd_{k-1}' \\ &+ 2dd_{k-2}^0 \end{aligned} \tag{2.11}$$

Example 2.1 When the bouquet B_2 is given co-roots u and v, one on each of its two self-loops, its partitioned double-root genus distribution is as shown in Table 2.2, and the result of self-amalgamation is the dipole D_4, as illustrated in Figure 2.4. We use Eq (2.11) to calculate the genus distribution of D_4.

Table 2.2: Double-root partials of B_2.

k	dd_k^0	ds_k^0	sd_k^0	ss_k^0	dd_k'	dd_k''	ds_k'	sd_k'	ss_k^1	ss_k^2	g_k
0	0	0	0	0	4	0	0	0	0	0	4
1	0	0	0	0	0	0	0	0	0	2	2

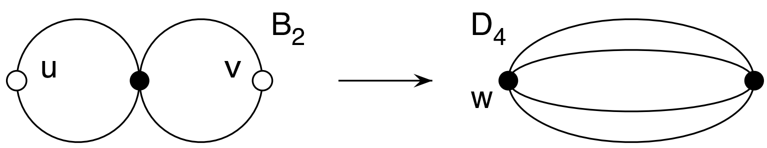

Figure 2.4: Self-amalgamation of bouquet B_2 yields dipole D_4.

In evaluating Eq (2.11), we can suppress terms on the right with negative subscripts or with a subscript greater than $\gamma_{\max}(B_2)$. Since the subpartials of types ds and sd are 0-valued for B_2, we can suppress them also.

$$\begin{aligned}
g_0(D_4) &= ss_1^2(B_2) + dd_0'(B_2) + 4dd_0''(B_2) + 6ss_0^1(B_2) + 5ss_0^2(B_2) \\
&= 2 + 4 + 0 + 0 + 0 \quad = \quad 6 \\
g_1(D_4) &= dd_1'(B_2) + 4dd_1''(B_2) + 6ss_1^1(B_2) + 5ss_1^2(B_2) \\
&\quad + 4dd_0^0(B_2) + 5dd_0'(B_2) + 2dd_0''(B_2) + 6ss_0^0(B_2) \\
&= 0 + 0 + 0 + 5 \cdot 2 + 0 + 5 \cdot 4 + 0 + 0 \quad = \quad 30
\end{aligned}$$

Example 1.1, redux We recall that the result of the self-amalgamation $*_{uv}(\ddot{K}_4, u, v)$ is the 4-wheel W_4, as illustrated in Figure 1.1. Table 2.3 gives the double-root partials of $(\ddot{K}_4, u, v)$, which we use in calculating the genus distribution of the 4-wheel W_4.

Table 2.3: Double-root partials of $\ddot{K}_4$.

k	dd_k^0	ds_k^0	sd_k^0	ss_k^0	dd_k'	dd_k''	ds_k'	sd_k'	ss_k^1	ss_k^2	g_k
0	2	0	0	0	0	0	0	0	0	0	2
1	0	0	0	0	0	4	4	4	0	2	14

In applying Eq (2.11), we again suppress terms on the right with out-of-range subscripts. We also suppress the 0-valued partials with subscript 0, that is, all except for dd_0^0. Moreover, to get all the summands for each genus onto one line, we also suppress mention of $\ddot{K}_4$ on the right.

$$\begin{aligned}
g_0(W_4) &= ss_1^2 \quad = \quad 2 \\
g_1(W_4) &= dd_1' + 4dd_1'' + 3ds_1' + 3sd_1' + 6ss_1^1 + 5ss_1^2 + 4dd_0^0 \\
&= 0 + 4 \cdot 4 + 3 \cdot 4 + 3 \cdot 4 + 0 + 5 \cdot 2 + 4 \cdot 2 \quad = \quad 58 \\
g_2(W_4) &= 4dd_1^0 + 5dd_1' + 2dd_1'' + 6ds_1^0 + 3ds_1' + 6sd_1^0 + 3sd_1' + 6ss_1^0 + 2dd_0^0 \\
&= 0 + 0 + 2 \cdot 4 + 0 + 3 \cdot 4 + 0 + 3 \cdot 4 + 0 + 2 \cdot 2 \quad = \quad 36
\end{aligned}$$

3 Amalgamating Two Doubly-Rooted Graphs

Toward our objective of deriving the genus distributions of a closed chain of copies of a graph, we now develop some production rules for obtaining double-rooted partials of the graph that results from amalgamating two double-rooted graphs. This will enable us to construct the double-rooted partials of arbitrarily long chains of copies of a given graph, in preparation for forming closed chains by merging the co-roots of the open chain.

Corresponding to the ten partials for a double-rooted graph, there are a hundred production rules. We presently confine ourselves to deriving the rules we need for specific calculations.

Doubly-Rooted Partials for a Doubled Path

In the present context, the ***doubled path*** DP_n is the graph obtained from an n-edge path P_n by doubling every edge. Since the only non-zero double-rooted partial for DP_1 is

$$dd''_0(DP_1) = 1$$

no table for DP_1 is needed. Figure 3.1 depicts the graph DP_3.

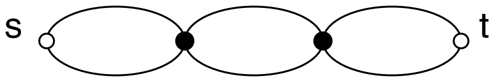

Figure 3.1: The doubled path DP_3.

Since the doubled path DP_1 will be the second amalgamand for each iteration, it is sufficient to derive the ten productions in which dd''_j is the second operand on the left side.

Theorem 3.1 *Let (G, s, t) and (H, u, v) be double-rooted graphs, both with 2-valent co-roots, and let $(W, s, v) = (G, s, t) * (H, u, v)$ be the amalgamation that merges vertices v and u. The following ten production rules hold:*

$$dd^0_i * dd''_j \longrightarrow 4dd^0_{i+j} + 2ds^0_{i+j+1} \tag{3.1}$$
$$ds^0_i * dd''_j \longrightarrow 6dd^0_{i+j} \tag{3.2}$$
$$sd^0_i * dd''_j \longrightarrow 4sd^0_{i+j} + 2ss^0_{i+j+1} \tag{3.3}$$
$$ss^0_i * dd''_j \longrightarrow 6sd^0_{i+j} \tag{3.4}$$
$$dd'_i * dd''_j \longrightarrow 2dd^0_{i+j} + 2dd'_{i+j} + 2ds'_{i+j+1} \tag{3.5}$$
$$dd''_i * dd''_j \longrightarrow 4dd'_{i+j} + 2ss^2_{i+j+1} \tag{3.6}$$
$$ds'_i * dd''_j \longrightarrow 6dd'_{i+j} \tag{3.7}$$
$$sd'_i * dd''_j \longrightarrow 2sd^0_{i+j} + 2sd'_{i+j} + 2ss^1_{i+j+1} \tag{3.8}$$
$$ss^1_i * dd''_j \longrightarrow 6sd'_{i+j} \tag{3.9}$$
$$ss^2_i * dd''_j \longrightarrow 4sd'_{i+j} + 2dd''_{i+j} \tag{3.10}$$

Proof The entire proof is by face-tracing on rotation projections. We supply drawings for three sample cases, easiest first.

Rule 3.6 Figure 3.2 illustrates the simplest configuration, corresponding to Rule (Rule 3.6). There are four amalgamated imbeddings like the drawing at the left, in which the two edge-ends from each graph are contiguous in the rotation at the merged vertex, and two imbeddings like the drawing at the right, in which edge-ends alternate.

In the former case, one strand (red) from graph G and one strand (blue) from graph H recombine into a single fb-walk that is incident on both roots of the amalgamated

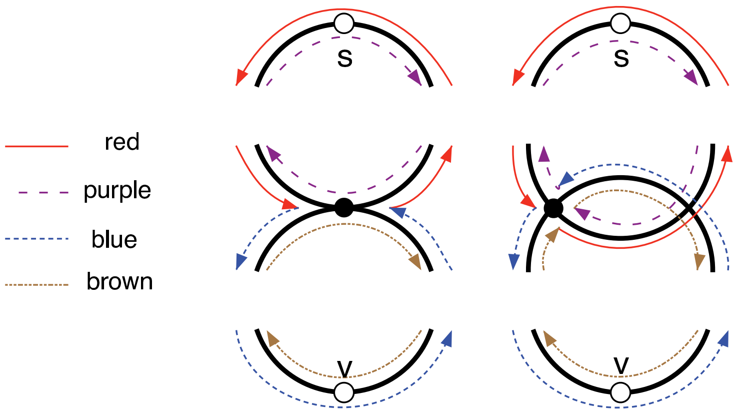

Figure 3.2: $dd''_i * dd''_j \longrightarrow 4dd'_{i+j} + 2ss^2_{i+j+1}$

graph. In the latter case, both fb-walks (red and purple) of G incident at t are broken into single strands, as are both strands (blue and brown) of H incident at u. Face-tracing establishes that the four strands recombine into a single f-b-walk that is incident on co-roots s and v of the merged graph. Face-tracing also establishes that when this fb-walk is broken at s into two strands, both strands are incident on co-root v. ♢ (Rule 3.6)

Rule 3.5 This follows from face-tracing on the rotation projections in variations of Figure 3.3. The drawing at the left, as shown, corresponds to an imbedding of type dd'. However, if the two edge-ends of vertex u were pasted to the other side of vertex t, then the imbedding would be of type dd^0. When the edge-ends at t and u alternate, as in the drawing on the right, either the blue strand or the brown strand recombines with the red strand, and the resulting imbedding is of type ds'. ♢ Rule (3.5)

Rule 3.10 Figure 3.4 illustrates the configuration for this rule. In the rotation projection at the right, the single fb-walk (red) at vertex s breaks into two strands, both of which are incident on vertex t. One red strand recombines with the blue strand of the other graph, and the other red strand recombines with the brown strand. The result is two fb-walks at s, both of which are incident on v. ♢ (Rule 3.10)

We leave proofs of the remaining rules to the reader. ♢ (Theorem 3.1)

Recursion for Double-Root Partials of a Doubled Path

For every partial that occurs in Theorem 3.1, we can derive the bivariate simultaneous recursion needed to calculate the genus distribution of every doubled path DP_n. We use the argument n in place of DP_n.

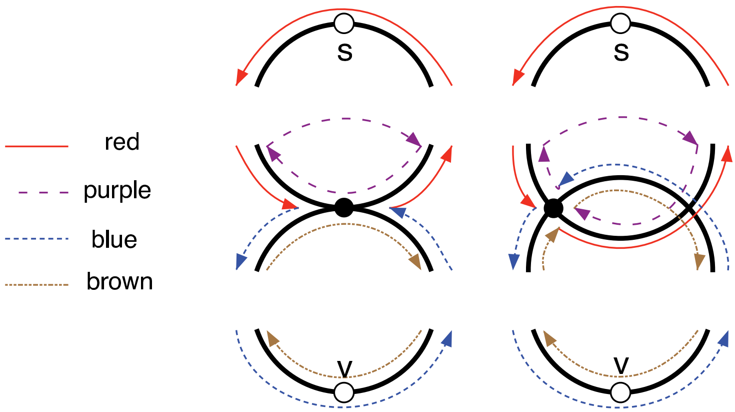

Figure 3.3: $dd'_i * dd''_j \longrightarrow 2dd^0_{i+j} + 2dd'_{i+j} + 2ds'_{i+j+1}$

Theorem 3.2 *The following simultaneous recursion holds for the doubled path graphs:*

$$dd^0_k(n) = 4dd^0_k(n-1) + 6ds^0_k(n-1) + 2dd'_k(n-1) \tag{3.11}$$
$$ds^0_k(n) = 2dd^0_{k-1}(n-1) \tag{3.12}$$
$$sd^0_k(n) = 4sd^0_k(n-1) + 6ss^0_k(n-1) + 2sd'_k(n-1) \tag{3.13}$$
$$ss^0_k(n) = 2sd^0_{k-1}(n-1) \tag{3.14}$$
$$dd'_k(n) = 4dd''_k(n-1) + 6ds'_k(n-1) + 2dd'_k(n-1) \tag{3.15}$$
$$dd''_k(n) = 2ss^2_k(n-1) \tag{3.16}$$
$$ds'_k(n) = 2dd'_{k-1}(n-1) \tag{3.17}$$
$$sd'_k(n) = 2sd'_k(n-1) + 6ss^1_k(n-1) + 4ss^2_k(n-1) \tag{3.18}$$
$$ss^1_k(n) = 2sd'_{k-1}(n-1) \tag{3.19}$$
$$ss^2_k(n) = 2dd''_{k-1}(n-1) \tag{3.20}$$

Proof For instance, Eq (3.11) for $dd^0_k(n)$ is derived by scanning the production rules of Theorem 3.1. We observe that a dd^0 term occurs in Rules (3.1), (3.2), and (3.5), into which we now substitute DP_{n-1} for G, DP_1 for H, and DP_n for W, thereby particularizing these three production rules.

$$\begin{aligned}
dd^0_i(n-1) * dd''_j(1) &\longrightarrow 4dd^0_{i+j}(n) + 2ds^0_{i+j+1}(n) \\
ds^0_i(n-1) * dd''_j(1) &\longrightarrow 6dd^0_{i+j}(n) \\
dd'_i(n-1) * dd''_j(1) &\longrightarrow 2dd^0_{i+j}(n) + 2dd'_{i+j}(n) + 2ds'_{i+j+1}(n)
\end{aligned}$$

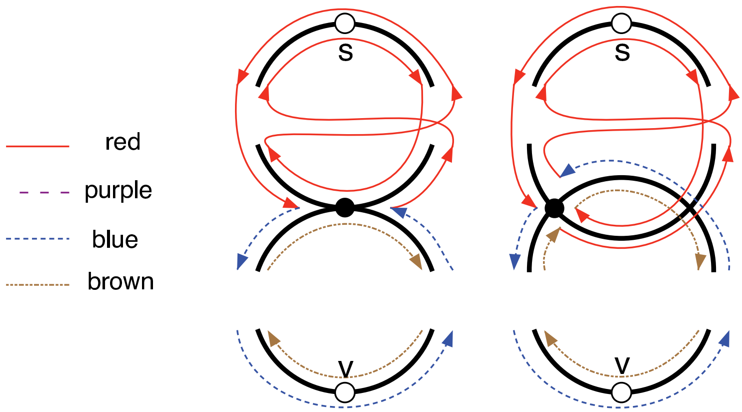

Figure 3.4: $ss_i^2 * dd_j'' \longrightarrow 4sd_{i+j}' + 2dd_{i+j}''$

Since the only imbedding of DP_1 is of genus 0, we have

$$dd_j''(1) \;=\; \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{otherwise} \end{cases}$$

It follows that

$$dd_k^0(n) \;=\; 4dd_k^0(n-1) + 6ds_k^0(n-1) + 2dd_k'(n-1)$$

Proofs of the remaining equations are left to the reader. (They can all be obtained at once by transposing an appropriate matrix.) ♢ (Theorem 3.2)

Remark The doubled path DP_n is homeomorphic to what Furst, Gross, and Statman [7] called the "cobblestone path" J_{n-1}. Accordingly, it has the same genus distribution. The benefit of the present more detailed derivation is that having the partitioned double-root genus distribution enables us to calculate the genus distribution for a "doubled cycle".

Table 3.1 below presents partitioned genus distributions for DP_1 to DP_5.

4 Distribution for a Closed Chain of Copies

With the use of a partitioned double-root genus distribution for an open chain of copies of a given graph, it is possible to calculate the genus distribution of a ***closed chain of copies*** of that graph, by which we mean the result of pasting together the co-roots of an open chain of copies of that graph.

Genus Distribution for a Doubled Cycle

By the ***doubled cycle*** DC_n, we mean the result of pasting together the two 2-valent vertices of the doubled path DP_n. We recognize that this is equivalent to a closed

Table 3.1: Double-root partials of doubled path DP_n.

	k	dd^0_k	ds^0_k	sd^0_k	ss^0_k	dd'_k	dd''_k	ds'_k	sd'_k	ss^1_k	ss^2_k	g_k
DP_1	0	0	0	0	0	0	1	0	0	0	0	1
DP_2	0	0	0	0	0	4	0	0	0	0	0	4
	1	0	0	0	0	0	0	0	0	0	2	2
DP_3	0	8	0	0	0	8	0	0	0	0	0	16
	1	0	0	0	0	0	4	8	8	0	0	20
DP_4	0	48	0	0	0	16	0	0	0	0	0	64
	1	0	16	16	0	64	0	16	16	0	0	128
	2	0	0	0	0	0	0	0	0	16	8	24
DP_5	0	224	0	0	0	32	0	0	0	0	0	256
	1	224	96	96	0	224	0	32	32	0	0	704
	2	0	0	0	32	0	16	128	128	32	0	336

chain of copies of the 2-cycle C_2. Figure 4.1 illustrates the doubled cycles DC_3, DC_4, and DC_5.

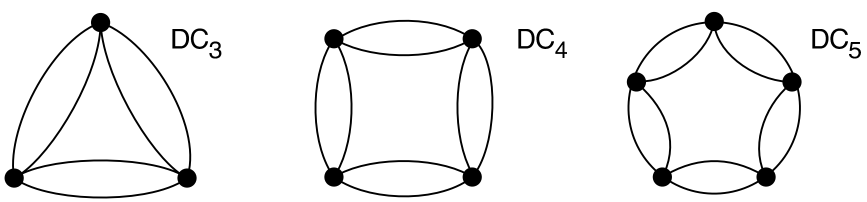

Figure 4.1: The doubled cycles DC_3, DC_4, and DC_5.

The genus distributions of doubled cycles $DC_1, \ldots, DC_5$ in Table 4.1 are calculated by applying Theorem 2.3 to Table 3.1.

Doubly-Rooted Partials for an Open Chain of Copies of $K_4 - e$

In [13], we gave a recursive specification of a sequence of single-rooted open chains X_n of copies of a graph X_1, and we derived recursions for the single-root partials of X_n. Our immediate task at hand is the construction of a recursion for the double-root partials of X_n, which we now construe to be double rooted.

$$\begin{aligned}(X_1,\, s_1,\, t_1) &= (K_4 - e,\, u,\, v) \text{ with two 2-valent co-roots}\\ (X_n,\, s_n,\, t_n) &= (X_{n-1}, s_{n-1}, t_{n-1}) * (K_4 - e,\, u,\, v) \text{ for } n > 1\end{aligned}$$

From those double-root partials, we will derive the genus distribution of a closed chain of copies of $K_4 - e$. The starting point is Table 4.2, which contains the double-root partials of $X_1 = K_4 - e$.

Table 4.1: Genus distribution of some doubled cycles DC_n.

	g_0	g_1	g_2	g_3
DC_1	4	2		
DC_2	6	30		
DC_3	8	136	72	
DC_4	16	440	840	
DC_5	32	1472	4832	1440

Table 4.2: Double-root partials of $(K_4 - e,\, u,\, v)$.

k	dd_k^0	ds_k^0	sd_k^0	ss_k^0	dd'_k	dd''_k	ds'_k	sd'_k	ss_k^1	ss_k^2	g_k
0	0	0	0	0	2	0	0	0	0	0	2
1	0	0	0	0	0	0	0	0	0	2	2

As we saw for the doubled cycle, the next step is to derive the set of productions with the non-zero partials of X_1 as second operand. Although there are only two partials needed as first operand in the derivation of partials for the genus distribution of $X_2 = X_1 * K_4 - e$, progressively more non-zero partials will emerge as the sequence grows. It so happens that none of the graphs X_n has a non-zero value for $dd''_k(X_n)$, so we omit productions with dd''_k from our set of production rules.

Theorem 4.1 *Let (G, s, t) and (H, u, v) be double-rooted graphs. The following production rules apply to deriving double-root partials of their amalgamation $(X, s, v) = (G, s, t) * (H, u, v)$.*

$$dd_i^0 * dd'_j \longrightarrow 4dd_{i+j}^0 + 2dd_{i+j+1}^0 \tag{4.1}$$
$$dd_i^0 * ss_j^2 \longrightarrow 4ds_{i+j}^0 + 2dd_{i+j}^0 \tag{4.2}$$
$$ds_i^0 * dd'_j \longrightarrow 6dd_{i+j}^0 \tag{4.3}$$
$$ds_i^0 * ss_j^2 \longrightarrow 6ds_{i+j}^0 \tag{4.4}$$
$$sd_i^0 * dd'_j \longrightarrow 4sd_{i+j}^0 + 2sd_{i+j+1}^0 \tag{4.5}$$
$$sd_i^0 * ss_j^2 \longrightarrow 4ss_{i+j}^0 + 2sd_{i+j}^0 \tag{4.6}$$
$$ss_i^0 * dd'_j \longrightarrow 6sd_{i+j}^0 \tag{4.7}$$
$$ss_i^0 * ss_j^2 \longrightarrow 6ss_{i+j}^0 \tag{4.8}$$
$$dd'_i * dd'_j \longrightarrow dd'_{i+j} + 3dd_{i+j}^0 + 2dd'_{i+j+1} \tag{4.9}$$

$$\begin{aligned}
dd'_i * ss^2_j &\longrightarrow 2ds'_{i+j} + 2ds^0_{i+j} + 2dd'_{i+j} && (4.10)\\
ds'_i * dd'_j &\longrightarrow 3dd^0_{i+j} + 3dd'_{i+j} && (4.11)\\
ds'_i * ss^2_j &\longrightarrow 6ds'_{i+j} && (4.12)\\
sd'_i * dd'_j &\longrightarrow 3sd^0_{i+j} + sd'_{i+j} + 2sd'_{i+j+1} && (4.13)\\
sd'_i * ss^2_j &\longrightarrow 2ss^1_{i+j} + 2ss^0_{i+j} + 2sd'_{i+j} && (4.14)\\
ss^1_i * dd'_j &\longrightarrow 3sd'_{i+j} + 3sd^0_{i+j} && (4.15)\\
ss^1_i * ss^2_j &\longrightarrow 6ss^1_{i+j} && (4.16)\\
ss^2_i * dd'_j &\longrightarrow 2sd'_{i+j} + 2sd^0_{i+j} + 2dd'_{i+j} && (4.17)\\
ss^2_i * ss^2_j &\longrightarrow 4ss^1_{i+j} + 2ss^2_{i+j} && (4.18)
\end{aligned}$$

Proof All of these production rules are derived by face-tracing. ♢

Lemma 4.2 *The production rules of Theorem 4.1 yield the following nine summations for the partials of the double-root amalgamation of the graphs* $(X_{n-1}, s_{n-1}, t_{n-1})$ *and* (X_1, u, v).

$$\begin{aligned}
dd^0_k(X_n) &= 4\sum_{i=0}^{k} dd^0_i(X_{n-1}) \cdot dd'_{k-i}(X_1) + 2\sum_{i=0}^{k-1} dd^0_i(X_{n-1}) \cdot dd'_{k-i-1}(X_1)\\
&\quad +2\sum_{i=0}^{k} dd^0_i(X_{n-1}) \cdot ss^2_{k-i}(X_1) + 6\sum_{i=0}^{k} ds^0_i(X_{n-1}) \cdot dd'_{k-i}(X_1)\\
&\quad +3\sum_{i=0}^{k} dd'_i(X_{n-1}) \cdot dd'_{k-i}(X_1) + 3\sum_{i=0}^{k} ds'_i(X_{n-1}) \cdot dd'_{k-i}(X_1) && (4.19)\\
ds^0_k(X_n) &= 4\sum_{i=0}^{k} dd^0_i(X_{n-1}) \cdot ss^2_{k-i}(X_1) + 6\sum_{i=0}^{k} ds^0_i(X_{n-1}) \cdot ss^2_{k-i}(X_1)\\
&\quad +2\sum_{i=0}^{k} dd'_i(X_{n-1}) \cdot ss^2_{k-i}(X_1) && (4.20)\\
sd^0_k(X_n) &= 4\sum_{i=0}^{k} sd^0_i(X_{n-1}) \cdot dd'_{k-i}(X_1) + 2\sum_{i=0}^{k-1} sd^0_i(X_{n-1}) \cdot dd'_{k-i-1}(X_1)\\
&\quad +2\sum_{i=0}^{k} sd^0_i(X_{n-1}) \cdot ss^2_{k-i}(X_1) + 6\sum_{i=0}^{k} ss^0_i(X_{n-1}) \cdot dd'_{k-i}(X_1)\\
&\quad +3\sum_{i=0}^{k} sd'_i(X_{n-1}) \cdot dd'_{k-i}(X_1) + 3\sum_{i=0}^{k} ss^1_i(X_{n-1}) \cdot dd'_{k-i}(X_1)\\
&\quad +2\sum_{i=0}^{k} ss^2_i(X_{n-1}) \cdot dd'_{k-i}(X_1) && (4.21)
\end{aligned}$$

$$ss_k^0(X_n) = 4\sum_{i=0}^{k} sd_i^0(X_{n-1}) \cdot ss_{k-i}^2(X_1) + 6\sum_{i=0}^{k} ss_i^0(X_{n-1}) \cdot ss_{k-i}^2(X_1) + 2\sum_{i=0}^{k} sd_i'(X_{n-1}) \cdot ss_{k-i}^2(X_1) \tag{4.22}$$

$$dd_k'(X_n) = \sum_{i=0}^{k} dd_i'(X_{n-1}) \cdot dd_{k-i}'(X_1) + 2\sum_{i=0}^{k-1} dd_i'(X_{n-1}) \cdot dd_{k-i-1}'(X_1) + 2\sum_{i=0}^{k} dd_i'(X_{n-1}) \cdot ss_{k-i}^2(X_1) + 3\sum_{i=0}^{k} ds_i'(X_{n-1}) \cdot dd_{k-i}'(X_1) + 2\sum_{i=0}^{k} ss_i^2(X_{n-1}) \cdot dd_{k-i}'(X_1) \tag{4.23}$$

$$ds_k'(X_n) = 2\sum_{i=0}^{k} dd_i'(X_{n-1}) \cdot ss_{k-i}^2(X_1) + 6\sum_{i=0}^{k} ds_i'(X_{n-1}) \cdot ss_{k-i}^2(X_1) \tag{4.24}$$

$$sd_k'(X_n) = \sum_{i=0}^{k} sd_i'(X_{n-1}) \cdot dd_{k-i}'(X_1) + 2\sum_{i=0}^{k-1} sd_i'(X_{n-1}) \cdot dd_{k-i-1}'(X_1) + 2\sum_{i=0}^{k} sd_i'(X_{n-1}) \cdot ss_{k-i}^2(X_1) + 3\sum_{i=0}^{k} ss_i^1(X_{n-1}) \cdot dd_{k-i}'(X_1) + 2\sum_{i=0}^{k} ss_i^2(X_{n-1}) \cdot dd_{k-i}'(X_1) \tag{4.25}$$

$$ss_k^1(X_n) = 2\sum_{i=0}^{k} sd_i'(X_{n-1}) \cdot ss_{k-i}^2(X_1) + 6\sum_{i=0}^{k} ss_i^1(X_{n-1}) \cdot ss_{k-i}^2(X_1) + 4\sum_{i=0}^{k} ss_i^2(X_{n-1}) \cdot ss_{k-i}^2(X_1) \tag{4.26}$$

$$ss_k^2(X_n) = 2\sum_{i=0}^{k} ss_i^2(X_{n-1}) \cdot ss_{k-i}^2(X_1) \tag{4.27}$$

Theorem 4.3 *The partials of the double-root amalgamation of the graphs* $(X_{n-1}, s_{n-1}, t_{n-1})$ *and* (X_1, u, v) *satisfy the following set of simultaneous recursions.*

$$dd_k^0(X_n) = 8dd_k^0(X_{n-1}) + 8dd_{k-1}^0(X_{n-1}) + 12ds_k^0(X_{n-1}) + 6dd_k'(X_{n-1}) + 6ds_k'(X_{n-1}) \tag{4.28}$$

$$ds_k^0(X_n) = 8dd_{k-1}^0(X_{n-1}) + 12ds_{k-1}^0(X_{n-1}) + 4dd_{k-1}'(X_{n-1}) \tag{4.29}$$

$$\begin{aligned} sd^0_k(X_n) &= 8sd^0_k(X_{n-1}) + 8sd^0_{k-1}(X_{n-1}) \\ &\quad +12ss^0_k(X_{n-1}) + 6sd'_k(X_{n-1}) \\ &\quad +6ss^1_k(X_{n-1}) + 2ss^2_k(X_{n-1}) \qquad (4.30) \\ ss^0_k(X_n) &= 8sd^0_{k-1}(X_{n-1}) + 12ss^0_{k-1}(X_{n-1}) \\ &\quad +4sd'_{k-1}(X_{n-1}) \qquad (4.31) \\ dd'_k(X_n) &= 2dd'_k(X_{n-1}) + 8dd'_{k-1}(X_{n-1}) \\ &\quad +6ds'_k(X_{n-1}) + 4ss^2_k(X_{n-1}) \qquad (4.32) \\ ds'_k(X_n) &= 4dd'_{k-1}(X_{n-1}) + 12ds'_{k-1}(X_{n-1}) \qquad (4.33) \\ sd'_k(X_n) &= 2sd'_k(X_{n-1}) + 8sd'_{k-1}(X_{n-1}) \\ &\quad +6ss^1_k(X_{n-1}) + 4ss^2_k(X_{n-1}) \qquad (4.34) \\ ss^1_k(X_n) &= 4sd'_{k-1}(X_{n-1}) + 12ss^1_{k-1}(X_{n-1}) \\ &\quad +8ss^2_{k-1}(X_{n-1}) \qquad (4.35) \\ ss^2_k(X_n) &= 4ss^2_{k-1}(X_{n-1}) \qquad (4.36) \end{aligned}$$

Proof The only non-zero partials of X_1 are $dd'_0(X_1) = 2$ and $ss^2_1(X_1) = 2$. Therefore, there is only one non-zero term in each of the summations of Lemma 4.2. ♢

Corollary 4.4 *The double-root partials for the graphs X_2 and X_3 are given by Table 4.3 and Table 4.4, respectively.*

Proof We apply the simultaneous recursions of Theorem 4.3 to Table 4.2 to obtain Table 4.3 and we then the apply the recursions to Table 4.3 to obtain Table 4.4. ♢

Table 4.3: Double-root partials of X_2.

k	dd^0_k	ds^0_k	sd^0_k	ss^0_k	dd'_k	dd''_k	ds'_k	sd'_k	ss^1_k	ss^2_k	g_k
0	12	0	0	0	4	0	0	0	0	0	16
1	0	8	8	0	24	0	8	8	0	0	56
2	0	0	0	0	0	0	0	0	16	8	24

Table 4.4: Double-root partials of X_3.

k	dd^0_k	ds^0_k	sd^0_k	ss^0_k	dd'_k	dd''_k	ds'_k	sd'_k	ss^1_k	ss^2_k	g_k
0	120	0	0	0	8	0	0	0	0	0	128
1	384	112	112	0	128	0	16	16	0	0	768
2	0	192	192	96	224	0	192	192	32	0	1120
3	0	0	0	0	0	0	0	0	256	32	288

Genus Distribution for a Closed Chain of Copies of $K_4 - e$

We can now calculate the genus distribution for a closed chain of n copies of $K_4 - e$, which we denote by CX_n, by applying Theorem 2.3 to the double-root partials of the graph X_n. Figure 4.2 illustrates a closed chain of 4 copies of $K_4 - e$. Table 4.5 gives the genus distributions of the first few graphs in this sequence.

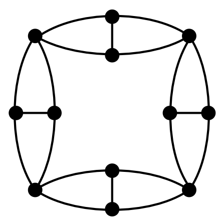

Figure 4.2: A closed chain of 4 copies of $K_4 - e$.

Table 4.5: Genus distribution of some closed chains of copies of $K_4 - e$.

	g_0	g_1	g_2	g_3	g_4
CX_1	4	20			
CX_2	4	148	424		
CX_3	8	744	5456	7616	
CX_4	16	4720	48384	154368	124288

5 Conclusions

The methods presented in this paper enable us to make the following types of calculations for genus distributions:

- a closed formula for the self-amalgamation $*_{uv}(G, u, v)$ of any double-rooted graph (G, u, v) with 2-valent roots whose double-root partitioned genus distribution is known;

- a closed formula for an iterated amalgamation of any finite sequence (repetitions allowed) of graphs (with 2-valent co-roots)

$$(G_1, u_1, v_1),\ (G_2, u_2, v_2),\ \ldots,\ (G_n, u_n, v_n)$$

  whose double-root partitioned genera are known.

- a recursion for any sequence of open chains of graphs formed by iterative amalgamation of copies of a given graph (G, u, v) with 2-valent co-roots and with a known double-root partitioned genus distribution;

- a recursion for any sequence of closed chains of graphs formed by iterative amalgamation of copies of a given graph (G, u, v) with 2-valent co-roots and with a known double-root partitioned genus distribution, and then completed (i.e., from an open chain to a closed chain) with a self-amalgamation.

That the potential usefulness of these enumerative methods goes well beyond the open and closed chains examined here is underscored by the fact that any Eulerian graph can be synthesized by iterative amalgamation of cycle graphs, which serve as building blocks with simple partitioned genus distributions.

References

[1] L. W. Beineke, R. J. Wilson, J. L. Gross and T. W. Tucker (eds.), *Topics in Topological Graph Theory*, Cambridge University Press, 2009.

[2] C. P. Bonnington, M. J. Grannell, T. S. Griggs and J. Širáň, Exponential families of non-isomorphic triangulations of complete graphs, *J. Combin. Theory Series B* **78** (2000), 169–184.

[3] C. P. Bonnington and C. H. C. Little, *The Foundations of Topological Graph Theory*, Springer, 1995.

[4] J. Chen, J. L. Gross and R. G. Rieper, Overlap matrices and total imbedding distributions, *Discrete Math.* **128** (1994), 73–94.

[5] Y. C. Chen, Y. P. Liu and T. Wang, The total embedding distributions of cacti and necklaces, *Acta Math. Sinica — English Series* **22** (2006), 1583–1590.

[6] M. Conder and P. Dobcsányi, Determination of all regular maps of small genus, *J. Combin. Theory Series B* **81** (2001), 224–242.

[7] M. L. Furst, J. L. Gross and R. Statman, Genus distribution for two classes of graphs, *J. Combin. Theory Series B* **46** (1989), 22–36.

[8] L. Goddyn, R. B. Richter and J. Širáň, Triangular embeddings of complete graphs from graceful labellings of paths, *J. Combin. Theory Series B* **97** (2007), 964–970.

[9] M. J. Grannell and T. S. Griggs, A lower bound for the number of triangular embeddings of some complete graphs and complete regular tripartite graphs, *J. Combin. Theory series B* **98** (2008), 637–650.

[10] J. L. Gross, Genus distributions of cubic outerplanar graphs, Preprint, 2010, 23 pp.

[11] J. L. Gross, Genus distribution of graphs under surgery: adding edges and splitting vertices, *New York J. Math.* **16** (2010), 161–178.

[12] J. L. Gross and M. L. Furst, Hierarchy for imbedding-distribution invariants of a graph, *J. Graph Theory* **11** (1987), 205–220.

[13] J. L. Gross, I. F. Khan and M. I. Poshni, Genus distribution of graph amalgamations: Pasting at root-vertices, *Ars Combin.* **94** (2010), 33–53.

[14] J. L. Gross, D. P. Robbins and T. W. Tucker, Genus distributions for bouquets of circles, *J. Combin. Theory Series B* **47** (1989), 292–306.

[15] J. L. Gross and T. W. Tucker, *Topological Graph Theory*, Dover, 2001; (original edn. Wiley, 1987).

[16] D. M. Jackson, Counting cycles in permutations by group characters with an application to a topological problem, *Trans. Amer. Math. Soc.* **299** (1987), 785–801.

[17] D. M. Jackson and T. I. Visentin, A character-theoretic approach to embeddings of rooted maps in an orientable surface of given genus, *Trans. Amer. Math. Soc.* **322** (1990), 343–363.

[18] D. M. Jackson and T. I. Visentin, *An Atlas of the Smaller Maps in Orientable and Nonorientable Surfaces*, Chapman & Hall/CRC Press, 2001.

[19] I. F. Khan, M. I. Poshni and J. L. Gross, Genus distribution of graph amalgamations: Pasting when one root has arbitrary degree, *Ars Math. Contemporanea* **3** (2010), 121–138.

[20] V. P. Korzhik and H-J Voss, Exponential families of non-isomorphic non-triangular orientable genus embeddings of complete graphs, *J. Combin. Theory Series B* **86** (2002), 86–211.

[21] J. H. Kwak and J. Lee, Genus polynomials of dipoles, *Kyungpook Math. J.* **33** (1993), 115–125.

[22] J. H. Kwak and J. Lee, Enumeration of graph embeddings, *Discrete Math.* **135** (1994), 129–151.

[23] J. H. Kwak and S. H. Shim, Total embedding distributions for bouquets of circles, *Discrete Math.* **248** (2002), 93–108.

[24] L. A. McGeoch, *Algorithms for two graph problems: computing maximum-genus imbedding and the two-server problem*, PhD thesis, Carnegie-Mellon University, 1987.

[25] B. Mohar and C. Thomassen, *Graphs on Surfaces*, Johns Hopkins Press, 2001.

[26] B. P. Mull, Enumerating the orientable 2-cell imbeddings of complete bipartite graphs, *J. Graph Theory* **30** (1999), 77–90.

[27] M. I. Poshni, I. F. Khan and J. L. Gross, Genus distribution of graphs under edge-amalgamations, *Ars Math. Contemporanea* **3** (2010), 69–86.

[28] R. G. Rieper, *The enumeration of graph embeddings*, PhD thesis, Western Michigan University, 1990.

[29] S. Stahl, Region distributions of graph embeddings and Stirling numbers, *Discrete Math.* **82** (1990), 57–78.

[30] S. Stahl, Permutation-partition pairs III: Embedding distributions of linear families of graphs, *J. Combin. Theory Series B* **52** (1991), 191–218.

[31] S. Stahl, Region distributions of some small diameter graphs, *Discrete Math.* **89** (1991), 281–299.

[32] E. H. Tesar, Genus distribution of Ringel ladders, *Discrete Math.* **216** (2000), 235–252.

[33] T. I. Visentin and S. W. Wieler, On the genus distribution of (p, q, n)-dipoles, *Electronic J. of Combin.* **14** (2007), No. R12.

[34] L. X. Wan and Y. P. Liu, Orientable embedding distributions by genus for certain types of graphs, *Ars Combin.* **79** (2006), 97–105.

[35] L. X. Wan and Y. P. Liu, Orientable embedding genus distribution for certain types of graphs, *J. Combin. Theory Series B* **47** (2008), 19–32.

[36] A. T. White, *Graphs of Groups on Surfaces*, North-Holland, 2001.

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