TOTAL EMBEDDING DISTRIBUTIONS OF CIRCULAR LADDERS

YICHAO CHEN, JONATHAN L. GROSS, AND TOUFIK MANSOUR

ABSTRACT. The *total embedding polynomial* of a graph G is the bivariate polynomial

$$\mathbb{I}_G(x,y) \;=\; \sum_{i=0}^{\infty} a_i x^i + \sum_{j=1}^{\infty} b_j y^j$$

where a_i is the number of embeddings, for $i = 0, 1, \ldots$, into the orientable surface S_i, and b_j is the number of embeddings, for $j = 1, 2, \ldots$, into the non-orientable surface N_j. The sequence $\{a_i(G)|i \geq 0\}\bigcup\{b_j(G)|j \geq 1\}$ is called the *total embedding distribution* of the graph G; it is known for relatively few classes of graphs, compared to the *genus distribution* $\{a_i(G)|i \geq 0\}$. The *circular ladder* graph CL_n is the Cartesian product $K_2 \Box C_n$ of the complete graph on two vertices and the cycle graph on n vertices. In this paper, we derive a closed formula for the total embedding distribution of circular ladders.

1. Introduction

Genus distributions of graphs have frequently been calculated in the past quarter-century, since the topic was inaugurated by Gross and Furst [7]. The contributions include [1, 6, 8, 10, 11, 14, 20, 21, 22, 23] and [25]. Quite recently, Gross [12] has derived a quadratic-time algorithm for any class of graphs of fixed treewidth and bounded degree, which yields a system of simultaneous recurrences, but no closed formulas. Total embedding distributions are known for somewhat fewer classes of graphs. Chen, Gross and Rieper [2] computed total embedding distributions for necklaces of type $(r, 0)$, closed-end ladders, and cobblestone paths. Kwak and Shim [15] computed it for bouquets of circles and dipoles. Also recently, Chen, Ou and Zou [4] obtained an explicit formula for the total embedding distributions of Ringel ladders.

Our concern in this paper is circular ladders. McGeoch [18] derived an explicit formula for the genus distributions of circular ladders and for Möbius ladders (and Li [16, 17] has re-calculated them). Yang and Liu [26] counted the embeddings of circular ladders and Möbius ladders in the projective plane and in the Klein bottle. In this paper, we derive an explicit formula for the total embedding distributions of circular ladders, with the aid of Mohar's overlap matrix [19] and of the Chebyshev polynomials of the second kind.

It is assumed that the reader is at least somewhat familiar with the basics of topological graph theory, as found in Gross and Tucker [9]. All graphs considered in this paper are connected. A *graph* $G = (V(G), E(G))$ is permitted to have loops and multiple edges. A *surface* is a compact 2-dimensional manifold, without boundary. In topology, surfaces are classified into

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the *orientable surfaces* S_g, with g handles ($g \geq 0$), and the *non-orientable surfaces* N_k, with k crosscaps ($k > 0$). The graph embeddings under discussion here are *cellular embeddings*. For any spanning tree of a graph G, the number of co-tree edges is called the *Betti number* of G, or the *cycle rank* of G, and is denoted by $\beta(G)$.

1.1. **Background.** A *rotation at a vertex* v of a graph G is a cyclic ordering of all the edge-ends (or equivalently, the half-edges) incident with v. A *pure rotation system* ρ of a graph G is the collection of rotations at all the vertices of G. An embedding of G into an orientable surface S induces a pure rotation system as follows:

> the rotation of the edge-ends at vertex v is the cyclic permutation corresponding to the order in which the edge-ends are encountered in an orientation-preserving tour around v.

Conversely, by the *Heffter-Edmonds principle*, every rotation system induces a unique embedding (up to homeomorphism) of G into some oriented surface S. The bijectivity of this correspondence implies that the total number of oriented embeddings is $\prod_{v\in G}(d_v-1)!$, where d_v is the degree of vertex v.

A *general rotation system* for a graph G is a pair (ρ, λ), where ρ is a pure rotation system and λ is a mapping $E(G) \to \{0, 1\}$. The edge e is said to be *twisted* (respectively, *untwisted*) if $\lambda(e) = 1$ (respectively, $\lambda(e) = 0$). It is well-known that every oriented embedding of a graph G can be described uniquely by a general rotation system (ρ, λ) with $\lambda(e) = 0$, for all $e \in E(G)$.

1.2. **Total embedding polynomial.** By allowing the parameter λ to take non-zero values, we can describe the non-orientable embeddings of a graph G. For any fixed spanning tree T, a *T-rotation system* (ρ, λ) of G is a general rotation system (ρ, λ) such that $\lambda(e) = 0$, for all $e \in E(T)$. Any two embeddings of G are considered to be the *same* if their T-rotation systems are combinatorially equivalent. Let Φ_G^T denote the set of all T-rotation systems of G. It is known that

$$|\Phi_G^T| \;=\; 2^{\beta(G)} \prod_{v\in V(G)} (d_v-1)!$$

This implies that the number of non-orientable embeddings of G is

$$(2^{\beta(G)}-1) \prod_{v\in V(G)} (d_v-1)!$$

Suppose that among these $|\Phi_G^T|$ embeddings of G, there are a_i embeddings, for $i = 0, 1, \ldots$, into the orientable surface S_i, and there are b_j embeddings, for $j = 1, 2, \ldots$, into the non-orientable surface N_j. We call the bivariate polynomial

$$\mathbb{I}_G^T(x, y) = \sum_{i=0}^{\infty} a_i x^i + \sum_{j=1}^{\infty} b_j y^j$$

the *T-distribution polynomial* of G.

It should be noted that the T-distribution polynomial is independent of the choice of spanning tree T. Thus, we may define the *total embedding polynomial* of G as the bivariate polynomial $\mathbb{I}_G(x, y) = \mathbb{I}_G^T(x, y)$, for any choice of a spanning tree T. We call the first and second parts of $\mathbb{I}_G(x, y)$ the *genus polynomial* of G and the *crosscap number polynomial* of G, respectively, and we denote them by $g_G(x) = \sum_{i=0}^{\infty} a_i x^i$ and $f_G(y) = \sum_{i=1}^{\infty} b_i y^i$, respectively. Thus, we have $\mathbb{I}_G(x, y) = g_G(x) + f_G(y)$.

1.3. **Overlap matrices.** Mohar [19] introduced an invariant that has been used numerous times in the calculation of graph embedding distributions, starting with [2]. Let T be a spanning tree of a graph G, and let (ρ, λ) be a T-rotation system. Let $e_1, e_2, \ldots, e_{\beta(G)}$ be the cotree edges of T. The *overlap matrix* of (ρ, λ) is the $\beta(G) \times \beta(G)$ matrix $M = [m_{ij}]$ over $\mathbb{Z}_2$ such that

$$m_{ij} = \begin{cases} 1, & \text{if } i = j \text{ and } e_i \text{ is twisted;} \\ 1, & \text{if } i \neq j \text{ and the restriction of the underlying pure rotation system} \\ & \quad \text{to the subgraph } T + e_i + e_j \text{ is nonplanar;} \\ 0, & \text{otherwise.} \end{cases}$$

When the restriction of the underlying pure rotation system to the subgraph $T + e_i + e_j$ is nonplanar, we say that edges e_i and e_j *overlap*. The power of the overlap matrix is indicated by this theorem.

Theorem 1.1 (Mohar [19])**.** *Let (ρ, λ) be a general rotation system for a graph, and let M be the overlap matrix with respect to any spanning tree. Then the rank of M equals twice the genus of the corresponding embedding surface, if that surface is orientable, and it equals the crosscap number otherwise. It is independent of the choice of a spanning tree.*

2. Overlap matrices of circular ladders

The *circular ladder* graph CL_n is the graph Cartesian product $C_n \times K_2$, where K_2 is the complete graph on two nodes and C_n is the cycle graph on n nodes. Figure 1 depicts the circular ladder graph CL_4.

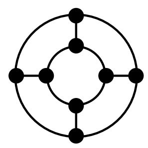

Figure 1. The circular ladder CL_4

2.1. **Gustin's represention of rotation systems for cubic graphs.** For drawing a planar representation of a rotation system on a cubic graph, we adopt the graphic "nomogram" (terminology due to Youngs) introduced by Gustin [13], and used extensively by Ringel and Youngs (see [24]) in their solution to the Heawood map-coloring problem. A trivalent vertex has two possible rotations. There are two possible cyclic orderings of each trivalent vertex. Under this convention, we color a vertex *black* if the rotation of the edge-ends incident on it is *clockwise*, and we color it *white* if the rotation is *counterclockwise*. We call any drawing of a graph that uses this convention to indicate a rotation system a *Gustin representation* of that rotation system.

In a Gustin nomogram, an edge is called *matched* if it has the same color at both endpoints; otherwise, it is called *unmatched*. In Figure 2, we have indicated our choice of a spanning tree for a generic circular ladder CL_{n+1} by thicker lines, so that the cotree edges are $e, f, e_1, e_2, \cdots, e_n$, and our partial choice of rotations at the vertices.

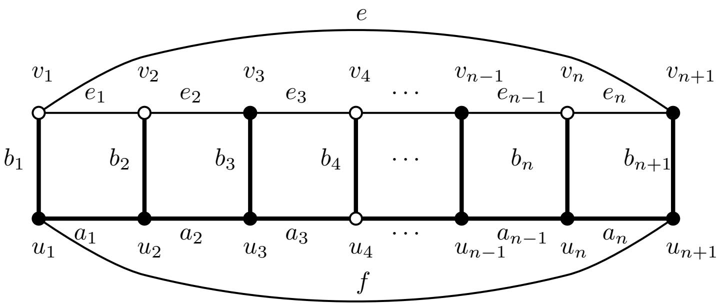

FIGURE 2. A Gustin nomogram for a circular ladder

2.2. **Overlap matrices for circular ladder graphs.** In the Gustin nomogram of Figure 2, we observe the following properties:

Property 2.1. *Two cotree edges e and e_i overlap if and only if the edge a_i is unmatched, for $i = 2, 3, \cdots, n-1$.*

Property 2.2. *Two cotree edges f and e_i overlap if and only if the edge a_i is unmatched, for $i = 1, 2, \cdots, n$.*

Property 2.3. *Two cotree edges e_i and e_{i+1} overlap if and only if the edge b_{i+1} is unmatched, for $i = 1, 2, \cdots, n-1$.*

Property 2.4. *The cotree edges e and e_1 overlap if and only if the vertices u_2 and v_1 are colored differently.*

Property 2.5. *The cotree edges e and e_n overlap if and only if the vertices u_n and v_{n+1} are colored differently.*

It follows that the overlap matrix of CL_{n+1} has this form, where $x_i, y_j, z_k \in \mathbb{Z}_2$:

$$M_{n+2}^{c,x,y,X,Y,Z} = \begin{pmatrix} x_e & c & x & z_2 & z_3 & \cdots & z_{n-1} & y \\ c & x_f & z_1 & z_2 & z_3 & \cdots & z_{n-1} & z_n \\ x & z_1 & x_1 & y_1 & & & & \\ z_2 & z_2 & y_1 & x_2 & y_2 & & \mathbf{0} & \\ z_3 & z_3 & & y_2 & x_3 & \ddots & & \\ \vdots & \vdots & & & \ddots & \ddots & y_{n-2} & \\ z_{n-1} & z_{n-1} & & \mathbf{0} & & y_{n-2} & x_{n-1} & y_{n-1} \\ y & z_n & & & & & y_{n-1} & x_n \end{pmatrix}$$

Note that $x_e = 1$ if and only if the edge e is twisted, and that $x_f = 1$ if and only if the edge f is twisted. Also note that

- $x_i = 1$ if and only if the edge e_i is twisted, for all $i = 1, 2, \ldots, n$;
- $y_j = 1$ if and only if b_{j+1} is unmatched, for all $j = 1, 2, \ldots, n-1$; and
- $z_k = 1$ if and only if a_k is unmatched, for all $k = 1, 2, \ldots, n$.

Moreover, we have $x = 1$ if and only if the colors of vertices v_1 and u_2 are different, and we have $y = 1$ if and only if the colors of vertices v_{n+1} and u_n are different. Furthermore, we have

Property 2.6. *The constant $c = 1$ (i.e., edges e and f overlap) if and only if the number of values of $z_1, z_2, \ldots, z_n$ equal to 1 is odd.*

Proof. The proof is by induction on n. Note that the edges e and f overlap if only if the vertices of u_1 and u_{n+1} are colored differently. For $n = 1$, this means the edge a_1 unmatched, this is equivalent to saying $z_1 = 1$.

Suppose that this is true for $n = k$ and the number of values of $z_1, z_2, \ldots, z_k$ equal to 1 is $2m + 1$, $(2m + 1 \leq n)$. It should be noted that the vertex set of $u_1, u_2, \ldots, u_{k+1}$ of CL_{k+1} can be obtained by inserting a vertex u between u_i and u_{i+1}, $i = 1, \ldots, k - 1$ of CL_k and relabel them. If the coloring of u_i and u_{i+1} are differently, no matter what assignment of colors to u, the number of values of $z_1, z_2, \ldots, z_{k+1}$ equal to 1 also equals $2m + 1$. Otherwise the coloring of u_i and u_{i+1} are the same (black or white), the number of values of $z_1, z_2, \ldots, z_{k+1}$ equal to 1 equals $2m + 1$ or $2m + 3$ according to the assignment of colors of u is the same or different from that of u_i and u_{i+1} respectively. □

Property 2.7. *For each fixed matrix of the form $M_{n+2}^{c,x,y,X,Y,Z}$, there are exactly two different T-rotation systems of the circular ladder CL_n for which $M_{n+2}^{c,x,y,X,Y,Z}$ is the overlap matrix.*

Proof. Given a matrix $M_{n+2}^{c,x,y,X,Y,Z}$, the values of $x, y, z_1, z_2, \cdots, z_n$ and $y_1, y_2, \cdots, y_{n-1}$ are determined.

$z_1 = 0$**:** If the vertex u_1 is black, then u_2 is also black, by Property 2.2. Alternatively, if the vertex u_1 is white, then the color of u_2 is also white. In either case, since the values of $x, y, z_2, \cdots, z_n$ and $y_1, y_2, \cdots, y_{n-1}$ are given, all the colors of $v_1, v_2, u_2, \cdots, v_{n+1}, u_{n+1}$ are determined, by Properties 2.2, 2.3, 2.4, 2.5, and 2.6. That is, all the rotations at vertices of CL_n are determined.

$z_1 = 1$**:** The proof is similar to the case $z_1 = 0$. Details are omitted. □

We define (with x and y ranging over $\mathbb{Z}_2$)

(1) $\mathcal{A}_{n+2}$ as the set of all matrices over $\mathbb{Z}_2$ of the form $M_{n+2}^{c,x,y,X,Y,Z}$;
(2) $\mathcal{A}_{n+2}(z) = \sum_{j=0}^{n+2} A_{n+2}(j)z^j$ as the *rank-distribution polynomial* of the set $\mathcal{A}_{n+2}$, where $A_{n+2}(j)$ denotes the number of matrices in A_{n+2} of rank j, that is, the number of overlap matrices of rank j for general rotation systems for the circular ladder CL_n;
(3) $\mathcal{B}_{n+2}$ as the set of all matrices of the form $M_{n+2}^{c,x,y,0,Y,Z}$; and
(4) $\mathcal{B}_{n+2}(z) = \sum_{j=0}^{n+2} B_{n+2}(j)z^j$ as the *rank-distribution polynomial* of the set $\mathcal{B}_{n+2}$, where $B_{n+2}(j)$ denotes the number of matrices in B_{n+2} of rank j, that is, the number of overlap matrices of rank j for pure rotation systems for the circular ladder CL_n.

It should be mentioned that the orientable case is precisely when X is identically 0.

In a matrix of the form $M_{n+2}^{c,x,y,X,Y,Z}$, suppose that we first add the second row to the first row and next add the second column to the first column. Without changing the rank of the matrix, this produces a matrix of the following form:

$$\begin{pmatrix}
x_e & c+x_f & x & 0 & \cdots & 0 & 0 & y \\
c+x_f & x_f & z_1 & z_2 & \cdots & z_{n-2} & z_{n-1} & z_n \\
x & z_1 & x_1 & y_1 & & & & \\
0 & z_2 & y_1 & x_2 & \ddots & & \mathbf{0} & \\
\vdots & \vdots & & \ddots & \ddots & & & \\
0 & z_{n-2} & & & & 0 & y_{n-2} & \\
0 & z_{n-1} & & \mathbf{0} & & y_{n-2} & x_{n-1} & y_{n-1} \\
y & z_n & & & & & y_{n-1} & x_n
\end{pmatrix}$$

For each single specific choice $xy \in \{00, 01, 10, 11\}$, we define

(1) $\mathcal{A}^{xy}_{n+2}$ as the set of all matrices over $\mathbb{Z}_2$ of the form $M^{c,x,y,X,Y,Z}_{n+2}$;
(2) $\mathcal{A}^{xy}_{n+2}(z) = \sum_{j=0}^{n+2} A^{xy}_{n+2}(j)z^j$ as the *rank-distribution polynomial* of the set $\mathcal{A}^{xy}_{n+2}$;
(3) $\mathcal{B}^{xy}_{n+2}$ as the set of all matrices of the form $M^{c,x,y,0,Y,Z}_{n+2}$; and
(4) $\mathcal{B}^{xy}_{n+2}(z) = \sum_{j=0}^{n+2} B^{xy}_{n+2}(j)z^j$ as the *rank-distribution polynomial* of the set $\mathcal{B}^{i}_{n+2}$.

Clearly, we have the following property.

Property 2.8. *For all* $n \geq 1$,

$$\mathcal{A}_{n+2}(z) = \sum_{xy=00,01,10,11} \mathcal{A}^{xy}_{n+2}(z) \quad \textit{and } \mathcal{B}_{n+2}(z) = \sum_{xy=00,01,10,11} \mathcal{B}^{xy}_{n+2}(z).$$

3. Rank-distribution polynomials of some ladder graphs

We recall that the *n-rung closed-end ladder* L_n can be obtained from the graphical cartesian product of an *n-vertex* path with the complete graph K_2 by doubling both its end edges. Figure 3 illustrates the 4-rung closed-end ladder L_4.

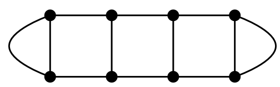

Figure 3. The 4-rung closed-end ladder L_4

We recall also that the *Ringel ladder* R_n, can be formed by subdividing the end-rungs of the closed-end ladder, L_n, and then adding an edge between these two new vertices. Figure 4 shows the Ringel ladder R_4.

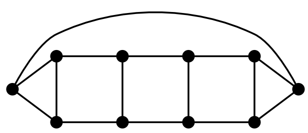

Figure 4. The Ringel ladder R_4

3.1. **Rank-distribution polynomial for Ringel ladders.** For the vectors $X = (x_0, x_1, \ldots, x_n)$, $Y = (y_1, y_2, \ldots, y_{n-1})$, and $Z = (z_1, z_2, \ldots, z_n)$, with $x_i, y_j, z_k \in \mathbb{Z}_2$, we define the matrices

$$M_n^{X,Y} = \begin{pmatrix} x_1 & y_1 & & & & & \\ y_1 & x_2 & y_2 & & & \mathbf{0} & \\ & y_2 & x_3 & y_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & \mathbf{0} & & y_{n-2} & x_{n-1} & y_{n-1} \\ & & & & y_{n-1} & x_n \end{pmatrix} \quad \text{and} \tag{1}$$

$$M_{n+1}^{X,Y,Z} = \begin{pmatrix} x_0 & z_1 & z_2 & z_3 & \cdots & z_{n-1} & z_n \\ z_1 & x_1 & y_1 & & & & \\ z_2 & y_1 & x_2 & y_2 & & \mathbf{0} & \\ z_3 & & y_2 & x_3 & \ddots & & \\ \vdots & & & \ddots & \ddots & y_{n-2} & \\ z_{n-1} & & \mathbf{0} & & y_{n-2} & x_{n-1} & y_{n-1} \\ z_n & & & & & y_{n-1} & x_n \end{pmatrix}. \tag{2}$$

As described by [2, 4], every overlap matrix of the closed-end ladder L_{n-1} has the form $M_{n+1}^{X,Y}$, and every overlap matrix of the Ringel ladder R_{n-1} has the form $M_{n+1}^{X,Y,Z}$. (Note that the subscripts of R_{n-1} and $M_{n+1}^{X,Y,Z}$ differ by two.) Now we further define

(1) $\mathcal{R}_n$ as the set of all matrices over $\mathbb{Z}_2$ of the form $M_n^{X,Y,Z}$;
(2) $C_n(j)$ as the number of overlap matrices for R_n that are of rank j;
(3) $\mathcal{R}_n(z) = \sum_{j=0}^{n+1} C_n(j) z^j$ as the *rank-distribution polynomial* of the set $\mathcal{R}_n$; and
(4) $\mathcal{L}_n(z)$ as the rank-distribution polynomial of the overlap matrices of the closed-end ladder L_{n-1}.

In the calculations of this section, we use the following two previously derived results:

Theorem 3.1. [3] *The polynomial $\mathcal{L}_n(z)$ satisfies the recurrence relation*

$$\mathcal{L}_n(z) = (1+2z)\mathcal{L}_{n-1}(z) + 4z^2\mathcal{L}_{n-2}(z)$$

with the initial conditions $\mathcal{L}_1(z) = 1 + z$ and $\mathcal{L}_2(z) = 4z^2 + 3z + 1$. Moreover, the generating function $\mathcal{L}(t; z) = \sum_{n\geq 1} \mathcal{L}_n(z) t^n$ is given by the formula

$$\mathcal{L}(t; z) = \frac{t(1 + z + 2z^2 t)}{1 - t - 2tz - 4z^2t^2}. \qquad \square$$

Theorem 3.2. [4] *The polynomial $\mathcal{R}_n(z)$ $(n \geq 3)$ satisfies the recurrence relation*

$$\mathcal{R}_{n+1}(z) = (4z+1)\mathcal{R}_n(z) + 16z^2\mathcal{R}_{n-1}(z) + 2^n z^2 \mathcal{L}_{n-1}(z).$$

with the initial condition $\mathcal{R}_2(z) = 4z^2 + 3z + 1$ and $\mathcal{R}_3(z) = 28z^3 + 28z^2 + 7z + 1$. Moreover, the generating function $\mathcal{R}(t; z) = \sum_{n\geq 2} \mathcal{R}_n(z) t^n$ is given by

$$\mathcal{R}(t; z) = \frac{t^2(1 + 3z + 4z^2 - 2(1 + 5z + 4z^2 + 2z^3)t - 16z^2(2 + 6z + 5z^2)t^2 - 128z^4(1 + z)t^3)}{(1 - 2t - 4tz - 16z^2t^2)(1 - t - 4tz - 16z^2t^2)}.$$

$\square$

Here we split the enumeration into even and odd parts:

(1) let $M_{n+1}^{X,Y,Z_{even}}$ be the number of matrices of the form $M_{n+1}^{X,Y,Z}$ and such that the sum of the elements of the vector Z is even;
(2) let $M_{n+1}^{X,Y,Z_{odd}}$ be the number of matrices of the form $M_{n+1}^{X,Y,Z}$ with $x_0 = 0$ and such that the sum of the elements of the vector Z is odd;
(3) let $M_{n+1}^{X,Y,Z_{even}}$ be the number of matrices of the form $M_{n+1}^{X,Y,Z}$ with $x_0 = 0$ and such that the sum of the elements of the vector Z is even;
(4) let $M_{n+1}^{X,Y,Z_{odd},1}$ be the number of matrices of that form with $x_0 = 1$ and such that the sum of the elements of Z is odd;
(5) let $\mathcal{R}_{n+1}^{E}(z)$ be the rank-distribution polynomial over the set $M_{n+1}^{X,Y,Z_{even}}$;
(6) let $\mathcal{R}_{n+1}^{O}(z)$ be the rank-distribution polynomial over the set $M_{n+1}^{X,Y,Z_{odd}}$;
(7) let $\mathcal{R}_{n+1}^{E,0}(z)$ be the rank-distribution polynomial over the set $M_{n+1}^{X,Y,Z_{even},0}$; and
(8) let $\mathcal{R}_{n+1}^{O,1}(z)$ be the rank-distribution polynomial over the set $M_{n+1}^{X,Y,Z_{odd},1}$.

Lemma 3.3. *The polynomial* $\mathcal{R}_n^{E,0}(z) + \mathcal{R}_n^{O,1}(z)$ $(n \geq 4)$ *satisfies the recurrence*

$$\mathcal{R}_{n+1}^{E,0}(z) + \mathcal{R}_{n+1}^{O,1}(z) \;=\; (2z+1)\left(\mathcal{R}_n^{E,0}(z) + \mathcal{R}_n^{O,1}(z)\right) + z\mathcal{R}_n(z) + 8z^2\mathcal{R}_{n-1}(z) + 2^{n-1}z^2\mathcal{L}_{n-1}(z)$$

with initial condition $\mathcal{R}_3^{E,0}(z) + \mathcal{R}_3^{O,1}(z) \;=\; 1 + 5z + 14z^2 + 12z^3$ *(where* $\mathcal{R}_{n-1}(z)$ *and* $\mathcal{L}_{n-1}(z)$ *are the rank-distribution polynomials of the overlap matrices of the Ringel ladder* R_{n-3} *and of the closed-end ladder* L_{n-2}*, respectively).*

Proof. We first prove the following claim.

Claim 1: The polynomial $\mathcal{R}_n^{E,0}(z)$ $(n \geq 4)$ satisfies the recurrence

$$\begin{aligned}\mathcal{R}_{n+1}^{E,0}(z) \;&= \mathcal{R}_n^{E,0}(z) + z\left(\mathcal{R}_n^{E,0}(z) + \mathcal{R}_n^{O,1}(z)\right) + z\mathcal{R}_n^{E}(z)\\ &\quad + 4z^2\mathcal{R}_{n-1}^{0}(z) + 2z^2\mathcal{R}_{n-1}(z) + 2^{n-2}z^2\mathcal{L}_{n-1}(z).\end{aligned}$$

with initial condition $\mathcal{R}_3^{E,0}(z) = 1 + 3z + 8z^2 + 4z^3$.

We analyze the form (see Equation (2)) of the overlap matrix $M_{n+1}^{X,Y,Z}$ into possible cases.

(1) Case 1: $x_n = 1$.
  - subcase 1: $y_{n-1} = 1$. We first add the last row to row n and then add the last column to column n. The resulting matrix has the following form.

$$\begin{pmatrix} 0 & z_1 & z_2 & \cdots & z_{n-2} & z_{n-1}+z_n & z_n \\ z_1 & x_1 & y_1 & & & & \\ z_2 & y_1 & x_2 & \ddots & & \mathbf{0} & \\ \vdots & & \ddots & \ddots & & & \\ z_{n-2} & & & & x_{n-2} & y_{n-2} & \\ z_{n-1}+z_n & & \mathbf{0} & & y_{n-2} & x_{n-1} & 0 \\ z_n & & & & & 0 & 1 \end{pmatrix}$$

    If $z_n = 0$, this matrix contributes a term $z\mathcal{R}_n^{E,0}(z)$ to the polynomial $\mathcal{R}_{n+1}^{E,0}(z)$. Otherwise $z_n = 1$, and it contributes a term $z\mathcal{R}_n^{E,1}(z)$
  - subcase 2: $y_{n-1} = 0$. As seen by a discussion similar to that for subcase 1, this subcase contributes a term $z\mathcal{R}_n^{E,0}(z) + z\mathcal{R}_n^{O,1}(z)$.

(2) Case 2: $x_n = 0$. This case, given by the following matrix, has four subcases.

$$
\begin{pmatrix}
0 & z_1 & z_2 & \cdots & z_{n-2} & z_{n-1} & z_n \\
z_1 & x_1 & y_1 & & & & \\
z_2 & y_1 & x_2 & \ddots & & \mathbf{0} & \\
\vdots & & \ddots & \ddots & & & \\
z_{n-2} & & & & x_{n-2} & y_{n-2} & \\
z_{n-1} & & \mathbf{0} & & y_{n-2} & x_{n-1} & 1 \\
z_n & & & & & 1 & 0
\end{pmatrix}
\tag{3}
$$

- subcase 1: $y_{n-1} = 1, z_n = 0$. No matter what the values of y_{n-2}, z_{n-1} and x_{n-1}, we can transform the above matrix to the following form, with no change of rank.

$$
\begin{pmatrix}
0 & z_1 & z_2 & \cdots & z_{n-2} & 0 & 0 \\
z_1 & x_1 & y_1 & & & & \\
z_2 & y_1 & x_2 & \ddots & & \mathbf{0} & \\
\vdots & & \ddots & \ddots & & & \\
z_{n-2} & & & & x_{n-2} & & \\
0 & & \mathbf{0} & & 0 & 0 & 1 \\
0 & & & & & 1 & 0
\end{pmatrix}
$$

There are four different combinations of values for the variables y_{n-2} and x_{n-1}. When $z_{n-1} = 0$, this matrix contributes $4z^2\mathcal{R}^{E,0}_{n-1}(z)$ to the polynomial $\mathcal{R}^{E,0}_{n+1}(z)$. When $z_{n-1} = 1$, it contributes $4z^2\mathcal{R}^{O,0}_{n-1}(z)$. Since $\mathcal{R}^{O,0}_{n-1}(z) + \mathcal{R}^{E,0}_{n-1}(z) = \mathcal{R}^{0}_{n-1}(z)$, this case contributes in all a term $4z^2\mathcal{R}^{0}_{n-1}(z)$.

- subcase 2: $y_{n-1} = 1, z_n = 1$. We add row n to the first row then add column n to the first column, as indicated by this matrix.

$$
\begin{pmatrix}
x_{n-1} & z_1 & z_2 & \cdots & z_{n-2} + y_{n-2} & z_{n-1} + x_{n-1} & 0 \\
z_1 & x_1 & y_1 & & & & \\
z_2 & y_1 & x_2 & \ddots & & \mathbf{0} & \\
\vdots & & \ddots & \ddots & & & \\
z_{n-2} + y_{n-2} & & & & x_{n-2} & y_{n-2} & \\
z_{n-1} + x_{n-1} & & \mathbf{0} & & y_{n-2} & x_{n-1} & 1 \\
0 & & & & & 1 & 0
\end{pmatrix}
$$

If $x_{n-1} = 0$, this case contributes $2z^2\mathcal{R}^{0}_{n-1}(z)$ to the polynomial $\mathcal{R}^{E,0}_{n+1}(z)$. Otherwise $x_{n-1} = 1$, and this case contributes $2z^2\mathcal{R}^{1}_{n-1}(z)$.

- subcase 3: $y_{n-1} = 0, z_n = 0$. This case also contributes $\mathcal{R}^{E,0}_{n}(z)$ to the polynomial $\mathcal{R}^{E,0}_{n+1}(z)$.

- subcase 4: $y_{n-1} = 0, z_n = 1$. No matter what values of $z_1, z_2, \ldots, z_{n-1}$ occur, we can transform Matrix (3) into the following form.

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & x_1 & y_1 & & & & \\ 0 & y_1 & x_2 & \ddots & & \mathbf{0} & \\ \vdots & & \ddots & \ddots & & & \\ 0 & & & & x_{n-2} & y_{n-2} & \\ 0 & & \mathbf{0} & & y_{n-2} & x_{n-1} & 0 \\ 1 & & & & & 0 & 0 \end{pmatrix}$$

Since there are 2^{n-2} possible choices of values for $z_1, z_2, \ldots, z_{n-1}$ with an even sum, this subcase contributes $2^{n-2}z^2\mathcal{L}_{n-1}(z)$ to the polynomial $\mathcal{R}^{E,0}_{n+1}(z)$.

By a similar analysis, we can substantiate a second claim.

Claim 2: The polynomial $\mathcal{R}^{O,1}_n(z)$ satisfies this recursion, for $(n \geq 4)$:

$$\begin{aligned} \mathcal{R}^{O,1}_{n+1}(z) &= \mathcal{R}^{O,1}_n(z) + z\left(\mathcal{R}^{O,1}_n(z) + \mathcal{R}^{E,0}_n(z)\right) + z\mathcal{R}^{O}_n(z) \\ &\quad + 4z^2\mathcal{R}^1_{n-1}(z) + 2z^2\mathcal{R}_{n-1}(z) + 2^{n-2}z^2\mathcal{L}_{n-1}(z). \end{aligned}$$

with initial conditions $\mathcal{R}^{O,1}_3(z) = 2z + 6z^2 + 8z^3$.

From the above two claims, the lemma follows. □

Proposition 3.4. *The generating function* $\mathcal{R}'(t;z) = \sum_{n\geq 3}(\mathcal{R}^{E,0}_n(z) + \mathcal{R}^{O,1}_n(z))t^n$ *has the closed form*

$$\frac{t^3 f(t;z)}{(1-2t-4tz-16z^2t^2)(1-t-4tz-16z^2t^2)(1-t-2tz)},$$

where

$$\begin{aligned} f(t;z) &= 14z^2 + 12z^3 + 1 + 5z + (-20z^4 - 3 - 84z^3 - 22z - 63z^2)t \\ &\quad + (2 + 20z + 34z^2 - 272z^4 - 272z^5 - 56z^3)t^2 - 16z^2(8z^4 - 43z^2 - 20z - 3 - 34z^3)t^3 \\ &\quad + 256z^4(1+2z)(1+z)^2t^4. \end{aligned}$$

Proof. Multiplying the recurrence relation of Lemma 3.3 for the polynomial $\mathcal{R}^{E,0}_n(z) + \mathcal{R}^{O,1}_n(z)$ by t^n and then summing over all $n \geq 3$, we establish that a closed form for $\mathcal{R}'(t;z)$ is given by

$$\frac{t^3 f(t;z)}{(1-2t-4tz-16z^2t^2)(1-t-4tz-16z^2t^2)(1-t-2tz)},$$

We have used the explicit formulas for the generating functions $\mathcal{R}(t;z)$ and $\mathcal{L}(t;z)$, as given in Theorem 3.1 and Theorem 3.2, respectively. □

3.2. **Expressing polynomials for circular ladders as a combination of polynomials for Ringel ladders and closed-end ladders.** We are now ready to approach the circular ladders.

Lemma 3.5. *The polynomial $\mathcal{A}_n^{01}(z)$ satisfies this equation for $n \geq 4$:*

$$\mathcal{A}_{n+2}^{01}(z) \;=\; (2z+1)\left(\mathcal{R}_{n+1}^{E,0}(z) + \mathcal{R}_{n+1}^{O,1}(z)\right) + 2^n z^2 \mathcal{L}_n(z) \tag{4}$$

where $\mathcal{R}_{n+1}(z)$ is the rank-distribution polynomial of the Ringel ladder R_{n-1} and $\mathcal{L}_n(z)$ is the rank-distribution polynomial of the closed-end ladder L_{n-1}.

Proof. There are two cases.

(1) Case 1: $x_e = 1$. Here the overlap matrix has the following form:

$$\begin{pmatrix} 1 & x_f+c & 0 & 0 & \cdots & 0 & 0 & 0 \\ x_f+c & x_f & z_1 & z_2 & \cdots & z_{n-2} & z_{n-1} & z_n \\ 0 & z_1 & x_1 & y_1 & & & & \\ 0 & z_2 & y_1 & x_2 & \ddots & & \mathbf{0} & \\ \vdots & \vdots & & \ddots & \ddots & & & \\ 0 & z_{n-2} & & & & x_{n-2} & y_{n-2} & \\ 0 & z_{n-1} & & \mathbf{0} & & y_{n-2} & x_{n-1} & y_{n-1} \\ 0 & z_n & & & & & y_{n-1} & x_n \end{pmatrix}$$

- subcase 1: $c = 0$. When $x_f = 0$, this subcase contributes a term $z\mathcal{R}_{n+1}^{E,0}(z)$; when $x_f = 1$, we add the first row to the second row, and then add the first column to the second column, to see that this subcase still contributes a term $z\mathcal{R}_{n+1}^{E,0}(z)$.
- subcase 2: $c = 1$. When $x_f = 1$, this subcase contributes a term $z\mathcal{R}_{n+1}^{O,1}(z)$; when $x_f = 0$, we first add the first row to the second row and then add the first column to the second column, to see that this case still contributes a term $z\mathcal{R}_{n+1}^{O,1}(z)$.

(2) Case 2: $x_e = 0$. In this case, the overlap matrix has the following form.

$$\begin{pmatrix} 0 & x_f+c & 0 & 0 & \cdots & 0 & 0 & 0 \\ x_f+c & x_f & z_1 & z_2 & \cdots & z_{n-2} & z_{n-1} & z_n \\ 0 & z_1 & x_1 & y_1 & & & & \\ 0 & z_2 & y_1 & x_2 & \ddots & & \mathbf{0} & \\ \vdots & \vdots & & \ddots & \ddots & & & \\ 0 & z_{n-2} & & & & x_{n-2} & y_{n-2} & \\ 0 & z_{n-1} & & \mathbf{0} & & y_{n-2} & x_{n-1} & y_{n-1} \\ 0 & z_n & & & & & y_{n-1} & x_n \end{pmatrix}$$

- subcase 1: $c = 0$. When $x_f = 0$, the contributed term is $\mathcal{R}_{n+1}^{E,0}(z)$. Otherwise $x_f = 1$; since there are 2^{n-1} choices of values of $z_1, z_2, \cdots, z_n$ with an even sum, the contributed term is $2^{n-1}z^2\mathcal{L}_n(z)$.
- subcase 2: $c = 1$. When $x_f = 0$, since there are 2^{n-1} choices of values of $z_1, z_2, \cdots, z_n$ with an odd sum, the contributed term is $2^{n-1}z^2\mathcal{L}_n(z)$; otherwise $x_f = 1$,, and the contributed term is $\mathcal{R}_{n+1}^{O,1}(z)$.

□

Lemma 3.6. *The polynomial* $\mathcal{A}_n^{10}(z)$ *satisfies this equation, for* $n \geq 4$*:*

$$\mathcal{A}_{n+2}^{10}(z) \;=\; z\mathcal{R}_{n+1}(z) + 8z^2\mathcal{R}_n(z). \tag{5}$$

where $\mathcal{R}_n(z)$ *is the rank-distribution polynomial of the Ringel ladder* R_{n-2}*.*

Proof. We examine the following two cases.

(1) Case 1: $x_e = 1$. In this case, the overlap matrix has the following form.

$$\begin{pmatrix} 1 & x_f+c & 1 & 0 & \cdots & 0 & 0 & 0 \\ x_f+c & x_f & z_1 & z_2 & \cdots & z_{n-2} & z_{n-1} & z_n \\ 1 & z_1 & x_1 & y_1 & & & & \\ 0 & z_2 & y_1 & x_2 & \ddots & & \mathbf{0} & \\ \vdots & \vdots & & \ddots & \ddots & & & \\ 0 & z_{n-2} & & & & x_{n-2} & y_{n-2} & \\ 0 & z_{n-1} & & \mathbf{0} & & y_{n-2} & x_{n-1} & y_{n-1} \\ 0 & z_n & & & & & y_{n-1} & x_n \end{pmatrix}$$

When we add the first row to the third row and add the first column to the third column, the resulting matrix has the following form.

$$\begin{pmatrix} 1 & x_f+c & 0 & 0 & \cdots & 0 & 0 & 0 \\ x_f+c & x_f & z_1+x_f+c & z_2 & \cdots & z_{n-2} & z_{n-1} & z_n \\ 0 & z_1+x_f+c & x_1 & y_1 & & & & \\ 0 & z_2 & y_1 & x_2 & \ddots & & \mathbf{0} & \\ \vdots & \vdots & & \ddots & \ddots & & & \\ 0 & z_{n-2} & & & & x_{n-2} & y_{n-2} & \\ 0 & z_{n-1} & & \mathbf{0} & & y_{n-2} & x_{n-1} & y_{n-1} \\ 0 & z_n & & & & & y_{n-1} & x_n \end{pmatrix}$$

- subcase 1: $c = 0$. If $x_f = 0$, the contributed term is $z\mathcal{R}_{n+1}^{E,0}(z)$. Otherwise $x_f = 1$, and we first add the first row to the second row, then add the first column to the second column, to establish that the contributed term is $z\mathcal{R}_{n+1}^{O,0}(z)$.
- subcase 2: $c = 1$. If $x_f = 0$, we first add the first row to the second row, and then add the first column to the second column. The contributed term is $z\mathcal{R}_{n+1}^{E,1}(z)$. Otherwise $x_f = 1$, and the contributed term is $z\mathcal{R}_{n+1}^{O,1}(z)$.

(2) Case 2: $x_e = 0$. In this case, the matrix has the following form.

$$\begin{pmatrix} 0 & x_f+c & 1 & 0 & \cdots & 0 & 0 & 0 \\ x_f+c & x_f & z_1 & z_2 & \cdots & z_{n-2} & z_{n-1} & z_n \\ 1 & z_1 & x_1 & y_1 & & & & \\ 0 & z_2 & y_1 & x_2 & \ddots & & \mathbf{0} & \\ \vdots & \vdots & & \ddots & \ddots & & & \\ 0 & z_{n-2} & & & & x_{n-2} & y_{n-2} & \\ 0 & z_{n-1} & & \mathbf{0} & & y_{n-2} & x_{n-1} & y_{n-1} \\ 0 & z_n & & & & & y_{n-1} & x_n \end{pmatrix}$$

- subcase 1: $c = 0$. First suppose that $x_f = 0$. There are four different choices of values for the variables x_1 and y_1. According to the values $z_1 = 0$ or $z_1 = 1$, this case contributes a term $4z^2\mathcal{R}_n^{E,0}(z)$ or $4z^2\mathcal{R}_n^{O,0}(z)$. For $x_f = 1$, we first add the third row to the second row and then add the third column to the second column. The resulting matrix has the following form.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1+x_1 & z_1+x_1 & z_2+y_1 & \cdots & z_{n-2} & z_{n-1} & z_n \\ 1 & z_1+x_1 & x_1 & y_1 & & & & \\ 0 & z_2+y_1 & y_1 & x_2 & \ddots & & \mathbf{0} & \\ \vdots & \vdots & & \ddots & \ddots & & & \\ 0 & z_{n-2} & & & & x_{n-2} & y_{n-2} & \\ 0 & z_{n-1} & & \mathbf{0} & & y_{n-2} & x_{n-1} & y_{n-1} \\ 0 & z_n & & & & & y_{n-1} & x_n \end{pmatrix}$$

No matter what the values of the variables of x_1, y_1 and z_1, we can transform that matrix into the following form.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1+x_1 & 0 & z_2+y_1 & \cdots & z_{n-2} & z_{n-1} & z_n \\ 1 & 0 & 0 & 0 & & & & \\ 0 & z_2+y_1 & 0 & x_2 & \ddots & & \mathbf{0} & \\ \vdots & \vdots & & \ddots & \ddots & & & \\ 0 & z_{n-2} & & & & x_{n-2} & y_{n-2} & \\ 0 & z_{n-1} & & \mathbf{0} & & y_{n-2} & x_{n-1} & y_{n-1} \\ 0 & z_n & & & & & y_{n-1} & x_n \end{pmatrix}$$

Since there are two choices of values for z_1, this case contributes a term $2z^2\mathcal{R}_n(z)$.

- subcase 2: $c = 1$. If $x_f = 0$, we first add the third row to the second row and then add the third column to the second column. Since the resulting matrix has the following form, the contributed term is $2z^2\mathcal{R}_n(z)$.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & x_1 & 0 & z_2+y_1 & \cdots & z_{n-2} & z_{n-1} & z_n \\ 1 & 0 & 0 & 0 & & & & \\ 0 & z_2+y_1 & 0 & x_2 & \ddots & & \mathbf{0} & \\ \vdots & \vdots & & \ddots & \ddots & & & \\ 0 & z_{n-2} & & & & x_{n-2} & y_{n-2} & \\ 0 & z_{n-1} & & \mathbf{0} & & y_{n-2} & x_{n-1} & y_{n-1} \\ 0 & z_n & & & & & y_{n-1} & x_n \end{pmatrix}$$

Otherwise $x_f = 1$. According to the values $z_1 = 0$ or $z_1 = 1$, this case contributes a term $4z^2\mathcal{R}_n^{E,1}(z)$ or $4z^2\mathcal{R}_n^{O,1}(z)$.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & z_1 & z_2 & \cdots & z_{n-2} & z_{n-1} & z_n \\ 1 & z_1 & x_1 & y_1 & & & & \\ 0 & z_2 & y_1 & x_2 & \ddots & & \mathbf{0} & \\ \vdots & \vdots & & \ddots & \ddots & & & \\ 0 & z_{n-2} & & & & x_{n-2} & y_{n-2} & \\ 0 & z_{n-1} & & \mathbf{0} & & y_{n-2} & x_{n-1} & y_{n-1} \\ 0 & z_n & & & & & y_{n-1} & x_n \end{pmatrix}$$

□

By symmetry, we infer this lemma also.

Lemma 3.7. *The polynomial $\mathcal{A}_n^{11}(z)$ $(n \geq 4)$ equals*

$$\mathcal{A}_{n+2}^{11}(z) = z\mathcal{R}_{n+1}(z) + 8z^2\mathcal{R}_n(z).$$

where $\mathcal{R}_n(z)$ is rank-distribution polynomial of Ringel ladders R_{n-2}.

We now define the form

$$M_{n+1}^{X,Y,Z,1} = \begin{pmatrix} x_0 & z_1 & z_2 & z_3 & \dots & z_{n-1} & z_n \\ z_1 & x_1 & y_1 & 0 & \dots & 0 & 1 \\ z_2 & y_1 & x_2 & y_2 & & & \\ z_3 & 0 & y_2 & x_3 & \ddots & \mathbf{0} & \\ \vdots & \vdots & & \ddots & \ddots & y_{n-2} & \\ z_{n-1} & & \mathbf{0} & & y_{n-2} & x_{n-1} & y_{n-1} \\ z_n & 1 & & & & y_{n-1} & x_n \end{pmatrix}.$$

and we define

(1) $\mathcal{H}_{n+1}$ as the set of all matrices over $\mathbb{Z}_2$ of the form $M_{n+1}^{X,Y,Z,1}$;
(2) $\mathcal{H}_{n+1}(z) = \sum_{j=0}^{n+1} H_{n+1}(j)z^j$ as the *rank-distribution polynomial* of the set $\mathcal{Q}_{n+1}$, where $H_{n+1}(j)$ is the number of matrices in $M_{n+1}^{X,Y,Z,1}$ has rank j;
(3) $M_{n+1}^{X,Y,Z_{even},1}$ as the number of matrices of the form $M_{n+1}^{X,Y,Z,1}$ in which the sum of vector Z is even;
(4) $M_{n+1}^{X,Y,Z_{odd},1}$ as the number of matrices of the form $M_{n+1}^{X,Y,Z,1}$ in which the sum of vector Z is odd;
(5) $\mathcal{H}_{n+1}^{E,0}(z)$ as the rank-distribution polynomial over the set $M_{n+1}^{X,Y,Z_{even},1}$ such that $x_0 = 0$; and
(6) $\mathcal{H}_{n+1}^{O,1}(z)$ as the rank-distribution polynomial over the set $M_{n+1}^{X,Y,Z_{odd},1}$ such that $x_0 = 1$.

Lemma 3.8. *For $n \geq 4$, the polynomial $\mathcal{H}_n^{E,0}(z) + \mathcal{H}_n^{O,1}(z)$ satisfies the recurrence*

$$\begin{aligned}\mathcal{H}_{n+1}^{E,0}(z) + \mathcal{H}_{n+1}^{O,1}(z) &= 2z\left(\mathcal{H}_n^{E,0}(z) + \mathcal{H}_n^{O,1}(z)\right) + 8z^2\left(\mathcal{H}_{n-1}^{O,1} + \mathcal{H}_{n-1}^{E,0}\right) \\ &\quad + z\mathcal{R}_n(z) + 12z^2\mathcal{R}_{n-1}(z).\end{aligned}$$

where $\mathcal{R}_n(z)$ is the rank-distribution polynomial of the Ringel ladder R_{n-2}. Moreover, we have the initial conditions $\mathcal{H}_2^{E,0}(z)+\mathcal{H}_2^{O,1}(z) = (1+z)^2$ and $\mathcal{H}_3^{E,0}(z)+\mathcal{H}_3^{O,1}(z) = 1+5z+14z^2+12z^3$.

Proof. We first prove the following property:

Claim 1: The polynomial $\mathcal{H}_n^{E,0}(z)$ $(n \geq 4)$ satisfies the recurrence

$$\begin{aligned}\mathcal{H}_{n+1}^{E,0}(z) &= z\left(\mathcal{H}_n^{E,0}(z)+\mathcal{H}_n^{O,1}\right)+z\mathcal{R}_n^{E}(z)+2z^2\left(\mathcal{R}_{n-1}^{O,0}(z)+\mathcal{R}_{n-1}^{E,1}\right)+2z^2\left(\mathcal{H}_{n-1}^{E,0}(z)+\mathcal{H}_{n-1}^{O,1}\right)\\ &\quad+4z^2\left(\mathcal{R}_{n-1}^{E,0}(z)+\mathcal{H}_{n-1}^{E,0}\right)+2z^2\mathcal{R}_{n-1}(z)+4z^2\mathcal{R}_{n-1}^{0}(z)\end{aligned}$$

where $\mathcal{R}_n(z)$ is the rank-distribution polynomial of the Ringel ladder R_{n-2}.

Here the overlap matrix has the following form. The discussion has two cases.

$$\begin{pmatrix} 0 & z_1 & z_2 & z_3 & \cdots & z_{n-1} & z_n \\ z_1 & x_1 & y_1 & 0 & \cdots & 0 & 1 \\ z_2 & y_1 & x_2 & y_2 & & & \\ z_3 & 0 & y_2 & x_3 & \ddots & \mathbf{0} & \\ \vdots & \vdots & & \ddots & \ddots & y_{n-2} & \\ z_{n-1} & & \mathbf{0} & & y_{n-2} & x_{n-1} & y_{n-1} \\ z_n & 1 & & & & y_{n-1} & x_n \end{pmatrix}$$

(1) Case 1: $x_n = 0$.
- subcase 1: $y_{n-1} = 0, z_n = 0$. No matter what values z_1, x_1, and y_1 take, we can transform the matrix above into the following form. Note that there are four different combinations of values for the variables y_1 and x_1.

$$\begin{pmatrix} 0 & 0 & z_2 & z_3 & \cdots & z_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ z_2 & 0 & x_2 & y_2 & & & \\ z_3 & 0 & y_2 & x_3 & \ddots & \mathbf{0} & \\ \vdots & \vdots & & \ddots & \ddots & y_{n-2} & \\ z_{n-1} & & \mathbf{0} & & y_{n-2} & x_{n-1} & 0 \\ 0 & 1 & & & & 0 & 0 \end{pmatrix}$$

  When $z_1 = 0$, it contributes a term $4z^2\mathcal{R}_{n-1}^{E,0}(z)$. When $z_1 = 1$, it contributes a term $4z^2\mathcal{R}_{n-1}^{O,0}(z)$.
- subcase 2: $y_{n-1} = 0, z_n = 1$. We add the second row to the first row, and then add the second column to the first column. The resulting matrix has the following form. Note that there are two different possible values for the variable z_1.

$$\begin{pmatrix} x_1 & z_1+x_1 & z_2+y_1 & z_3 & \cdots & z_{n-1} & 0 \\ z_1+x_1 & x_1 & y_1 & 0 & \cdots & 0 & 1 \\ z_2+y_1 & y_1 & x_2 & y_2 & & & \\ z_3 & 0 & y_2 & x_3 & \ddots & \mathbf{0} & \\ \vdots & \vdots & & \ddots & \ddots & y_{n-2} & \\ z_{n-1} & & \mathbf{0} & & y_{n-2} & x_{n-1} & 0 \\ 0 & 1 & & & & 0 & 0 \end{pmatrix}$$

  When $x_1 = 0$, it contributes a term $2z^2\mathcal{R}_{n-1}^{0}(z)$. When $x_1 = 1$, it contributes a term $2z^2\mathcal{R}_{n-1}^{1}(z)$.

- subcase 3: $y_{n-1} = 1, z_n = 0$. We add row n to the second row, and then add column n to the second column. The resulting matrix has the following form.

$$\begin{pmatrix}
0 & z_1+z_{n-1} & z_2 & z_3 & \cdots & z_{n-2} & z_{n-1} & 0 \\
z_1+z_{n-1} & x_1 & y_1 & 0 & \cdots & y_{n-2} & x_{n-1} & 0 \\
z_2 & y_1 & x_2 & y_2 & & & & \\
z_3 & 0 & y_2 & x_3 & \ddots & & \mathbf{0} & \\
\vdots & \vdots & & \ddots & \ddots & y_{n-3} & & \\
z_{n-2} & y_{n-2} & & & y_{n-3} & x_{n-2} & y_{n-2} & \\
z_{n-1} & x_{n-1} & \mathbf{0} & & & y_{n-2} & x_{n-1} & 1 \\
0 & 0 & & & & & 1 & 0
\end{pmatrix}$$

No matter what assignments of the variables of z_{n-1}, x_{n-1}, and y_{n-2}, we can transform the matrix immediately above to the following form. Note that there are four different choices of the variables z_{n-1} and x_{n-1}.

$$\begin{pmatrix}
0 & z_1+z_{n-1} & z_2 & z_3 & \cdots & z_{n-2} & 0 & 0 \\
z_1+z_{n-1} & x_1 & y_1 & 0 & \cdots & y_{n-2} & 0 & 0 \\
z_2 & y_1 & x_2 & y_2 & & & & \\
z_3 & 0 & y_2 & x_3 & \ddots & & \mathbf{0} & \\
\vdots & \vdots & & \ddots & \ddots & y_{n-3} & & \\
z_{n-2} & y_{n-2} & & & y_{n-3} & x_{n-2} & 0 & \\
0 & 0 & \mathbf{0} & & & 0 & 0 & 1 \\
0 & 0 & & & & & 1 & 0
\end{pmatrix}$$

When $y_{n-2} = 0$, it contributes a term $4z^2\mathcal{R}^{E,0}_{n-1}(z)$. When $y_{n-2} = 1$, it contributes a term $4z^2\mathcal{H}^{E,0}_{n-1}(z)$.

- subcase 4: $y_{n-1} = 1, z_n = 1$. We add row n to the first and second rows, and then add column n to the first and second columns. The resulting matrix has the following form.

$$\begin{pmatrix}
x_{n-1} & z_1+z_{n-1}+x_{n-1} & z_2 & z_3 & \cdots & z_{n-2}+y_{n-2} & z_{n-1}+x_{n-1} & 0 \\
z_1+z_{n-1}+x_{n-1} & x_1 & y_1 & 0 & \cdots & y_{n-2} & x_{n-1} & 0 \\
z_2 & y_1 & x_2 & y_2 & & & & \\
z_3 & 0 & y_2 & x_3 & \ddots & & \mathbf{0} & \\
\vdots & \vdots & & \ddots & \ddots & y_{n-3} & & \\
z_{n-2}+y_{n-2} & y_{n-2} & & & y_{n-3} & x_{n-2} & y_{n-2} & \\
z_{n-1}+x_{n-1} & x_{n-1} & \mathbf{0} & & & y_{n-2} & x_{n-1} & 1 \\
0 & 0 & & & & & 1 & 0
\end{pmatrix}$$

For any combination of values of z_{n-1}, x_{n-1}, and y_{n-2}, we can transform the above matrix to the following form. Note that there are two possible values for z_{n-1}.

$$\begin{pmatrix} x_{n-1} & z_1+z_{n-1}+x_{n-1} & z_2 & z_3 & \dots & z_{n-2}+y_{n-2} & 0 & 0 \\ z_1+z_{n-1}+x_{n-1} & x_1 & y_1 & 0 & \dots & y_{n-2} & 0 & 0 \\ z_2 & y_1 & x_2 & y_2 & & & & \\ z_3 & 0 & y_2 & x_3 & \ddots & & \mathbf{0} & \\ \vdots & \vdots & & \ddots & \ddots & y_{n-3} & & \\ z_{n-2}+y_{n-2} & y_{n-2} & & & y_{n-3} & x_{n-2} & 0 & \\ 0 & 0 & \mathbf{0} & & & 0 & 0 & 1 \\ 0 & 0 & & & & & 1 & 0 \end{pmatrix}$$

When $y_{n-2}=0$, depending whether $x_{n-1}=0$ or $x_{n-1}=1$, it contributes $2z^2\mathcal{R}^{O,0}_{n-1}(z)$ or $2z^2\mathcal{R}^{E,1}_{n-1}(z)$. When $y_{n-2}=1$, it contributes $2z^2\mathcal{H}^{O,0}_{n-1}(z)$ or $2z^2\mathcal{H}^{E,1}_{n-1}(z)$.

(2) Case 2: $x_n=1$. There are four subcases.

- subcase 1: $y_{n-1}=0, z_n=0$. We add the last row to the second row, and then add the last column to the second column. The resulting matrix has the following form. This subcase contributes $z\mathcal{R}^{E,0}_n(z)$.

$$\begin{pmatrix} 0 & z_1 & z_2 & z_3 & \dots & z_{n-1} & 0 \\ z_1 & x_1 & y_1 & 0 & \dots & 0 & 0 \\ z_2 & y_1 & x_2 & y_2 & & & \\ z_3 & 0 & y_2 & x_3 & \ddots & \mathbf{0} & \\ \vdots & \vdots & & \ddots & \ddots & y_{n-2} & \\ z_{n-1} & & \mathbf{0} & & y_{n-2} & x_{n-1} & 0 \\ 0 & 0 & & & & 0 & 1 \end{pmatrix}$$

- subcase 2: $y_{n-1}=0, z_n=1$. We add the last row to the first and second rows, and then add the last column to the first and second columns. The resulting matrix has the following form. This subcase also contributes a term $z\mathcal{R}^{E,1}_n(z)$.

$$\begin{pmatrix} 1 & z_1+1 & z_2 & z_3 & \dots & z_{n-1} & 0 \\ z_1+1 & x_1 & y_1 & 0 & \dots & 0 & 0 \\ z_2 & y_1 & x_2 & y_2 & & & \\ z_3 & 0 & y_2 & x_3 & \ddots & \mathbf{0} & \\ \vdots & \vdots & & \ddots & \ddots & y_{n-2} & \\ z_{n-1} & & \mathbf{0} & & y_{n-2} & x_{n-1} & 0 \\ 0 & 0 & & & & 0 & 1 \end{pmatrix}$$

- subcase 3: $y_{n-1}=1, z_n=0$. We add the last row to rows 2 and n, and then add the last column to columns 2 and n. The resulting matrix has the following form.

This subcase contributes $z\mathcal{H}_n^{E,0}(z)$.

$$\begin{pmatrix} 0 & z_1 & z_2 & z_3 & \dots & z_{n-1} & 0 \\ z_1 & x_1 & y_1 & 0 & \dots & 1 & 0 \\ z_2 & y_1 & x_2 & y_2 & & & \\ z_3 & 0 & y_2 & x_3 & \ddots & \mathbf{0} & \\ \vdots & \vdots & & \ddots & \ddots & y_{n-2} & \\ z_{n-1} & 1 & \mathbf{0} & & y_{n-2} & x_{n-1} & 0 \\ 0 & 0 & & & & 0 & 1 \end{pmatrix}$$

- subcase 4: $y_{n-1} = 1, z_n = 1$. We add the last row to rows 1, 2, and n, and then add the last column to the columns 1, 2, and n. The resulting matrix has the following form. This subcase contributes $z\mathcal{H}_n^{O,1}(z)$.

$$\begin{pmatrix} 1 & z_1+1 & z_2 & z_3 & \dots & z_{n-1}+1 & 0 \\ z_1+1 & x_1 & y_1 & 0 & \dots & 1 & 0 \\ z_2 & y_1 & x_2 & y_2 & & & \\ z_3 & 0 & y_2 & x_3 & \ddots & \mathbf{0} & \\ \vdots & \vdots & & \ddots & \ddots & y_{n-2} & \\ z_{n-1}+1 & 1 & \mathbf{0} & & y_{n-2} & x_{n-1} & 0 \\ 0 & 0 & & & & 0 & 1 \end{pmatrix}$$

In a similar way, we can establish this second claim:

Claim 2: The polynomial $\mathcal{H}_n^{O,1}(z)$ $(n \geq 4)$ satisfies the recurrence

$$\begin{aligned}\mathcal{H}_{n+1}^{O,1}(z) =& z\left(\mathcal{H}_n^{E,0}(z) + \mathcal{H}_n^{O,1}\right) + z\mathcal{R}_n^{O}(z) + 2z^2\left(\mathcal{R}_{n-1}^{O,0}(z) + \mathcal{R}_{n-1}^{E,1}\right) + 2z^2\left(\mathcal{H}_{n-1}^{E,0}(z) + \mathcal{H}_{n-1}^{O,1}\right) \\ &+ 4z^2\left(\mathcal{R}_{n-1}^{O,1}(z) + \mathcal{H}_{n-1}^{O,1}\right) + 2z^2\mathcal{R}_{n-1}(z) + 4z^2\mathcal{R}_{n-1}^{1}(z)\end{aligned}$$

where $\mathcal{R}_n(z)$ is the rank-distribution polynomial of the Ringel ladder R_{n-2}.

The above two claims imply the theorem. □

Proposition 3.9. *The generating function* $\mathcal{H}'(t;z) = \sum_{n\geq 3}(\mathcal{H}_n^{E,0}(z) + \mathcal{H}_n^{O,1}(z))t^n$ *is given by*

$$\frac{t^2 f(t;z)}{(1-2t-4tz-16z^2t^2)(1-t-4tz-16z^2t^2)(1+2zt)(1-4zt)},$$

where

$$\begin{aligned}f(t;z) = {}& (1+z)^2 - (2+9z^2+11z-2z^3)t - (1+25z^2+20z^4+66z^3)t^2 \\ &+ (2-96z^5+46z^2+72z^4+16z+136z^3)t^3 + 16z^2(29z^2-4z^4+3+26z^3+14z)t^4 \\ &+ 256z^4(1+z)^2t^5.\end{aligned}$$

Proof. We multiply the recurrence relation of Lemma 3.8 for the polynomial $\mathcal{H}_n^{E,0}(z) + \mathcal{H}_n^{O,1}(z)$ by t^n and we sum over all $n \geq 3$ to deduce that the generating function $\mathcal{H}'(t;z)$ is given by

$$\frac{t^2 f(t;z)}{(1-2t-4tz-16z^2t^2)(1-t-4tz-16z^2t^2)(1+2zt)(1-4zt)},$$

where we used the closed formula of Theorem 3.1 for the generating function $\mathcal{R}(t;z)$. □

Lemma 3.10. *The polynomial* $\mathcal{A}_n^{11}(z)$ $(n \geq 4)$ *satisfies the equation*

$$\mathcal{A}_{n+2}^{11}(z) = 2z\left(\mathcal{H}_{n+1}^{E,0}(z) + \mathcal{H}_{n+1}^{O,1}\right) + 8z^2\left(\mathcal{H}_n^{E,0}(z) + \mathcal{H}_n^{O,1}(z)\right) + 4z^2\mathcal{R}_n(z). \tag{6}$$

where $\mathcal{R}_{n+1}(z)$ *is the rank-distribution polynomial of the Ringel ladder* R_{n-1} *and* $\mathcal{L}_{n-1}(z)$ *is the rank-distribution polynomial of the closed-end ladder* L_{n-2}.

Proof. There are two cases.

(1) $x_e = 1$. We add the first row to rows 3 and $n+2$, and then add the first column to columns 3 and $n+2$. The resulting matrix has the following form.

$$\begin{pmatrix} 1 & c+x_f & 0 & 0 & \cdots & 0 & 0 & 0 \\ c+x_f & x_f & z_1+c+x_f & z_2 & \cdots & z_{n-2} & z_{n-1} & z_n+c+x_f \\ 0 & z_1+c+x_f & x_1 & y_1 & & & & 1 \\ 0 & z_2 & y_1 & x_2 & \ddots & & \mathbf{0} & \\ \vdots & \vdots & & \ddots & \ddots & & & \\ 0 & z_{n-2} & & & & x_{n-2} & y_{n-2} & \\ 0 & z_{n-1} & & \mathbf{0} & & y_{n-2} & x_{n-1} & y_{n-1} \\ 0 & z_n+c+x_f & 1 & & & & y_{n-1} & x_n \end{pmatrix}$$

In the subcase $c + x_f = 0$, we have either $c = x_f = 0$ or $c = x_f = 1$. This subcase contributes the terms $z\mathcal{H}_{n+1}^{E,0}(z)$ and $z\mathcal{H}_{n+1}^{O,1}(z)$. In the subcase $c + x_f = 1$, we have either $c = 0, x_f = 1$ or $c = 1, x_f = 0$. Here we add the first row to the second row and then add the first column to the second column. The resulting matrix has the following form, and this subcase contributes the terms $z\mathcal{H}_{n+1}^{E,0}(z)$ and $z\mathcal{H}_{n+1}^{O,1}(z)$.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & x_f+1 & z_1+1 & z_2 & \cdots & z_{n-2} & z_{n-1} & z_n+1 \\ 0 & z_1+1 & x_1 & y_1 & & & & 1 \\ 0 & z_2 & y_1 & x_2 & \ddots & & \mathbf{0} & \\ \vdots & \vdots & & \ddots & \ddots & & & \\ 0 & z_{n-2} & & & & x_{n-2} & y_{n-2} & \\ 0 & z_{n-1} & & \mathbf{0} & & y_{n-2} & x_{n-1} & y_{n-1} \\ 0 & z_n+1 & 1 & & & & y_{n-1} & x_n \end{pmatrix}$$

(2) $x_e = 0$. We add the last row to the third row, and then add the last column to the third column. The resulting matrix has the following form.

$$\begin{pmatrix} 0 & c+x_f & 0 & 0 & \cdots & 0 & 0 & 1 \\ c+x_f & x_f & z_1+z_n & z_2 & \cdots & z_{n-2} & z_{n-1} & z_n \\ 0 & z_1+z_n & x_1 & y_1 & & & y_{n-1} & x_n \\ 0 & z_2 & y_1 & x_2 & \ddots & & \mathbf{0} & \\ \vdots & \vdots & & \ddots & \ddots & & & \\ 0 & z_{n-2} & & & & x_{n-2} & y_{n-2} & \\ 0 & z_{n-1} & y_{n-1} & \mathbf{0} & & y_{n-2} & x_{n-1} & y_{n-1} \\ 1 & z_n & x_n & & & & y_{n-1} & x_n \end{pmatrix}$$

- subcase 1: $c + x_f = 0$. No matter what the values of z_n, x_n and y_{n-1}, the matrix immediately above can be transformed into the following form.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & x_f & z_1+z_n & z_2 & \cdots & z_{n-2} & z_{n-1} & 0 \\ 0 & z_1+z_n & x_1 & y_1 & & & y_{n-1} & 0 \\ 0 & z_2 & y_1 & x_2 & \ddots & & \mathbf{0} & \\ \vdots & \vdots & & \ddots & \ddots & & & \\ 0 & z_{n-2} & & & & x_{n-2} & y_{n-2} & \\ 0 & z_{n-1} & y_{n-1} & \mathbf{0} & & y_{n-2} & x_{n-1} & 0 \\ 1 & 0 & 0 & & & & 0 & 0 \end{pmatrix}$$

When $y_{n-1} = 0$, according to whether $c = x_f = 0$ or $c = x_f = 1$, this subcase contributes $4z^2\mathcal{R}_n^{E,0}(z)$ or $4z^2\mathcal{R}_n^{O,1}(z)$. When $y_{n-1} = 1$, it contributes $4z^2\mathcal{H}_n^{E,0}(z)$ or $4z^2\mathcal{H}_n^{O,1}(z)$.

- subcase 2: $c + x_f = 1$. We add the last row to the first and second rows, and we then add the last column to the first and second columns. The resulting matrix has the following form.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & x_f & z_1+z_n+x_n & z_2 & \cdots & z_{n-2} & z_{n-1}+y_{n-1} & z_n+x_n \\ 0 & z_1+z_n+x_n & x_1 & y_1 & & & y_{n-1} & x_n \\ 0 & z_2 & y_1 & x_2 & \ddots & & \mathbf{0} & \\ \vdots & \vdots & & \ddots & \ddots & & & \\ 0 & z_{n-2} & & & & x_{n-2} & y_{n-2} & \\ 0 & z_{n-1}+y_{n-1} & y_{n-1} & \mathbf{0} & & y_{n-2} & x_{n-1} & y_{n-1} \\ 1 & z_n+x_n & x_n & & & & y_{n-1} & x_n \end{pmatrix}$$

For any values of z_n, x_n and y_{n-1}, the above matrix can be transformed into the following form. When $y_{n-1} = 0$, it contributes $2z^2\mathcal{R}_n(z)$. When $y_{n-1} = 1$, it contributes $2z^2\mathcal{H}_n(z)$.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & x_f & z_1+z_n+x_n & z_2 & \cdots & z_{n-2} & z_{n-1}+y_{n-1} & 0 \\ 0 & z_1+z_n+x_n & x_1 & y_1 & & & y_{n-1} & 0 \\ 0 & z_2 & y_1 & x_2 & \ddots & & \mathbf{0} & \\ \vdots & \vdots & & \ddots & \ddots & & & \\ 0 & z_{n-2} & & & & x_{n-2} & y_{n-2} & \\ 0 & z_{n-1}+y_{n-1} & y_{n-1} & \mathbf{0} & & y_{n-2} & x_{n-1} & 0 \\ 1 & 0 & 0 & & & & 0 & 0 \end{pmatrix}$$

□

4. The total embedding polynomial of circular ladders

We recall that the *Chebyshev polynomials of the second kind* are defined by the *Chebyshev recurrence system*:

$$\begin{aligned} U_0(t) &= 1 \\ U_1(t) &= 2t \\ U_n(t) &= 2tU_{n-1}(t) - U_{n-2}(t) \end{aligned} \tag{7}$$

Theorem 4.1. *We have* $\sum_{n\geq 4} g_{CL_n}(x)t^n = \frac{2}{t}\mathcal{B}(t;\sqrt{x})$*. Moreover, for all* $n \geq 4$*, the number of distinct cellular imbeddings of* CL_n *in a surface of genus* j *is*

$$\begin{cases} \frac{7n+j}{j}2^{3j-3}\binom{n-j-1}{j-1} + \frac{n-4j+2}{j}2^{3j-3}\binom{n-j+1}{j-1} + \frac{n}{j-1}2^{n+j-1}\binom{n-j}{j-2} & \\ \quad +2^{n-1}\delta_{n,2j+2} + 2^n\delta_{n,2j+1} - 3\cdot 2^{n-1}\delta_{n,2j} & j \geq 2, \\ 2^n + 8n - 2 + 8\delta_{n,4} & j = 1, \\ 2 & j = 0. \end{cases}$$

where δ *is the Kronecker delta function.*

Proof. A derivation of the formula given here is driven by [27]. It is not hard to check that our formula is equivalent to the original formula of [18, Theorem 3.10]. □

Corollary 4.2. [27] *For all* $n \geq 4$,

$$\begin{aligned} g_{CL_n}(x) &= 1 - x + \frac{1-3x-2\sqrt{x}}{4x}(-2\sqrt{x})^n + \frac{1-3x+2\sqrt{x}}{4x}(2\sqrt{x})^n \\ &+ 2^n x(i\sqrt{2x})^n\left[U_n\left(\frac{1}{2i\sqrt{2x}}\right) - U_{n-2}\left(\frac{1}{2i\sqrt{2x}}\right)\right] \\ &+ (1-x)(2i\sqrt{2x})^n\left[U_n\left(\frac{1}{4i\sqrt{2x}}\right) - U_{n-2}\left(\frac{1}{4i\sqrt{2x}}\right)\right], \end{aligned}$$

where U_s *is the* s^{th} *Chebyshev polynomial of the second kind and* $i^2 = -1$

Now we can prove our main result.

Theorem 4.3. *The generating function* $\mathcal{A}(t;z) = \sum_{n\geq 2}\mathcal{A}_n(z)t^n$ *is given by*

$$\frac{t^2 f(t;z)}{(1-2t-4tz-16z^2t^2)(1-t-4tz-16z^2t^2)(1-t-2tz)(1+2tz)(1-4tz)},$$

where

$$\begin{aligned} f(t;z) &= (1+z)^2 + 2(z-1)(6z^2+6z+1)t - (67z^2+9z+162z^3+2+68z^4)t^2 \\ &+ (141z^2+282z^3+5+8z^4+44z-384z^5)t^3 \\ &+ (1424z^4+2z^2-2+704z^6+424z^3+2128z^5-24z)t^4 \\ &+ 32z^2(6z+1)(16z^4+7z^3-5z^2-6z-2)t^5 - 128z^4(5+58z^3+32z+69z^2)t^6 \\ &- 2048z^6(1+2z)(1+z)^2t^7. \end{aligned}$$

Proof. By Property 2.8, Lemmas 3.5, 3.6, 3.7, and 3.10 together with Lemmas 3.3 and 3.8 we obtain

$$\mathcal{A}_n(z) \;=\; \mathcal{R}_n^{E,0}(z)+\mathcal{R}_n^{O,1}(z)+\mathcal{H}_n^{E,0}(z)+\mathcal{H}_n^{O,1}(z).$$

By multiplying by t^n and summing over all $n\geq 2$, we have

$$\mathcal{A}(t;z)=\mathcal{H}'(t;z)+\mathcal{R}'(t;z),$$

where the generating functions $\mathcal{H}'(t;z)$ and $\mathcal{R}'(t;z)$ are given by Propositions 3.4 and 3.9, respectively. □

Theorem 4.4. *For all $n\geq 4$, the total genus polynomial of circular ladders CL_n is as follows:*

$$\mathbb{I}_{CL_n}(x,y) \;=\; 2\mathcal{B}_{n+1}(\sqrt{x})+2\mathcal{A}_{n+1}(y)-2\mathcal{B}_{n+1}(y).$$

The generating function $ICL(t;x,y)=\sum_{n\geq 3}\mathbb{I}_{CL_n}(x,y)t^n$ is given by

$$\begin{aligned}ICL(t;x,y) \;=\; \frac{2}{t}\Big(&\mathcal{B}(t;\sqrt{x})+\mathcal{A}(t;y)-(1+y)^2t^2-(2+24y^3+10y+28y^2)t^3\\ &-(1+11y+80y^2+212y^3+208y^4)t^4-\mathcal{B}(t;y)\Big).\end{aligned}$$

Moreover, for all $n\geq 4$,

$$\begin{aligned}\mathbb{I}_{CL_n}(x,y)=y^2-x&+\frac{1-3x-2\sqrt{x}}{4x}(-2\sqrt{x})^n+\frac{1-3x+2\sqrt{x}}{4x}(2\sqrt{x})^n\\
&+x(2i\sqrt{2x})^n\beta_n\left(\frac{1}{2i\sqrt{2x}}\right)+(1-x)(2i\sqrt{2x})^n\beta_n\left(\frac{1}{4i\sqrt{2x}}\right)\\
&-\frac{(y^2-1)(6y+1)(4y)^{n-1}}{3y}-\frac{(4y-1)(3y-1)(y+1)(-2y)^{n-2}}{3}+(1-y)(1+2y)^n\\
&+(3y+1)(y-1)(2y)^{n-2}+2(y^2-1)(2\sqrt{2}iy)^n\alpha_n\left(\frac{1}{4\sqrt{2}iy}\right)-2y^2(2\sqrt{2}iy)^n\alpha_n\left(\frac{1}{2\sqrt{2}iy}\right)\\
&+2(1-y)(1+2y)(4iy)^n\alpha_n\left(\frac{1+4y}{8iy}\right)+4y^2(4iy)^n\alpha_n\left(\frac{1+2y}{4iy}\right)\end{aligned}$$

where U_s is the s-th Chebyshev polynomial of the second kind, $i^2=-1$,

$$\alpha_n(t)=U_n(t)-tU_{n-1}(t) \text{ and } \beta_n(t)=U_n(t)-U_{n-2}(t).$$

Proof. By Property 2.7, we have $\mathbb{I}_{CL_n}(x,y)=2\mathcal{B}_{n+1}(\sqrt{x})+2\mathcal{A}_{n+1}(y)-2\mathcal{B}_{n+1}(y)$. Multiplying by t^n and summing over all $n\geq 4$ with using Theorems 4.1 and 4.3, we obtain

$$\begin{aligned}ICL(t;x,y)=\frac{2}{t}\Big(&\mathcal{B}(t;\sqrt{x})+\mathcal{A}(t;y)-(1+y)^2t^2-(2+24y^3+10y+28y^2)t^3\\ &-(1+11y+80y^2+212y^3+208y^4)t^4-\mathcal{B}(t;y)\Big).\end{aligned}$$

Clearly, the coefficient of t^n, $n \geq 4$, in $\frac{2}{t}\mathcal{B}(t;\sqrt{x})$ is given by Corollary 4.2. Now we consider the coefficient of t^n in

$$f = \frac{2}{t}\Big(\mathcal{A}(t;y) - (1+y)^2t^2 - (2+24y^3+10y+28y^2)t^3 - (1+11y+80y^2+212y^3+208y^4)t^4 - \mathcal{B}(t;y)\Big).$$

Rewriting f by partial fraction decomposition, we obtain

$$\begin{aligned} f &= -2(196y^3+212y^2+61y+11)yt^3 - 4(12y^2+11y+5)yt^2 - 4(y+2)yt - y^2 \\ &- \frac{y^2-2y^3+2y-1}{4y^2} + \frac{y^2-1}{1-t} + \frac{1+6y-6y^3-y^2}{12y^2(1-4ty)} + \frac{3y^2-2y-1}{4y^2(1-2ty)} - \frac{12y^3+5y^2-6y+1}{12y^2(1+2ty)} \\ &+ \frac{1-y}{1-t-2ty} + \frac{2y^2-2-y^2t+t}{1-t-8y^2t^2} - \frac{2y^2(1-t)}{1-2t-8y^2t^2} + \frac{2+2y-4y^2+8y^3t-t-2y^2t-5ty}{1-t-4ty-16y^2t^2} \\ &+ \frac{4y^2(1-t-2ty)}{1-2t-4ty-16y^2t^2}. \end{aligned}$$

Let $n \geq 4$, then the coefficient of t^n in f is given by

$$\begin{aligned} [t^n]f = [t^n]\Big\{ &\frac{y^2-1}{1-t} + \frac{1+6y-6y^3-y^2}{12y^2(1-4ty)} + \frac{3y^2-2y-1}{4y^2(1-2ty)} - \frac{12y^3+5y^2-6y+1}{12y^2(1+2ty)} \\ &+ \frac{1-y}{1-t-2ty} + \frac{2y^2-2-y^2t+t}{1-t-8y^2t^2} - \frac{2y^2(1-t)}{1-2t-8y^2t^2} \\ &+ \frac{2+2y-4y^2+8y^3t-t-2y^2t-5ty}{1-t-4ty-16y^2t^2} + \frac{4y^2(1-t-2ty)}{1-2t-4ty-16y^2t^2}.\Big\}, \end{aligned}$$

which is equivalent to

$$\begin{aligned} [t^n]f &= y^2 - 1 - \frac{(y^2-1)(6y+1)(4y)^{n-1}}{3y} - \frac{(4y-1)(3y-1)(y+1)(-2y)^{n-2}}{3} + (1-y)(1+2y)^n \\ &+ (3y+1)(y-1)(2y)^{n-2} + 2(y^2-1)(2\sqrt{2}iy)^n\alpha_n\left(\frac{1}{4\sqrt{2}iy}\right) \\ &- 2y^2(2\sqrt{2}iy)^n\alpha_n\left(\frac{1}{2\sqrt{2}iy}\right) + 2(1-y)(1+2y)(4iy)^n\alpha_n\left(\frac{1+4y}{8iy}\right) + 4y^2(4iy)^n\alpha_n\left(\frac{1+2y}{4iy}\right), \end{aligned}$$

which completes the proof. □

For instance our theorem for $n = 4, 5, 6, 7$ gives

$$\begin{aligned} \mathbb{I}_{CL_4}(x,y) &= 2+54x+24y+200x^2+1288y^3+3264y^4+192y^2+3168y^5 \\ \mathbb{I}_{CL_5}(x,y) &= 2+70x+30y+320x^3+632x^2+2560y^3+11240y^4+23232y^6+282y^2+27168y^5 \\ \mathbb{I}_{CL_6}(x,y) &= 2+110x+36y+2656x^3+1328x^2+4740y^3+27360y^4+169856y^7+211840y^6 \\ &\quad +424y^2+105936y^5 \\ \mathbb{I}_{CL_7}(x,y) &= 2+182x+42y+3584x^4+9984x^3+2632x^2+8652y^3+60368y^4+1663488y^7 \\ &\quad +933408y^6+618y^2+1208832y^8+302512y^5 \end{aligned}$$

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[27] http://www.cs.columbia.edu/∼gross/

College of Mathematics and Econometrics, Hunan University, 410082 Changsha, China

E-mail address: ycchen@hnu.edu.cn

Department of Computer Science, Columbia University, New York, NY 10027 USA

E-mail address: gross@cs.columbia.edu

Department of Mathematics, University of Haifa, 31905 Haifa, Israel

E-mail address: tmansour@univ.haifa.ac.il