COMBINATORIAL CONJECTURES THAT IMPLY LOCAL LOG-CONCAVITY OF GRAPH GENUS POLYNOMIALS

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Abstract. The 25-year old *LCGD Conjecture* is that the genus distribution of every graph is log-concave. We present herein a new topological conjecture, called the *Local Log-Concavity Conjecture*. We also present a purely combinatorial conjecture, which we prove to be equivalent to the Local Log-Concavity Conjecture. We use the equivalence to prove the Local Log-Concavity Conjecture for graphs of maximum degree four. We then show how a formula of David Jackson could be used to prove log-concavity for the genus distributions of various partial rotation systems, with straight-forward application to proving the *local log-concavity* of additional classes of graphs. We close with an additional conjecture, whose proof, along with proof of the Local Log-Concavity Conjecture, would affirm the LCGD Conjecture.

1. Introduction

This paper presents several conjectures related to the phenomenon of *local log-concavity*. All of the imbedding surfaces of concern here are oriented.

The number of imbeddings of a graph G in the oriented surface S_i is denoted by $g_i(G)$ or g_i. Then the ***genus distribution polynomial*** (or, simply, ***graph genus polynomial***) of G is

$$\Gamma_G(z) = \sum_{i=\gamma_{\min}(G)}^{\gamma_{\max}(G)} g_i z^i,$$

where $\gamma_{\min}(G)$ and $\gamma_{\max}(G)$ are the minimum genus and maximum genus of G, respectively. All graphs are taken to be connected, all

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surfaces are taken to be closed and oriented, and all imbeddings are taken to be cellular.

In our graphs, vertices in figures and commentary are labeled by lower case letters and edges are labeled by integers. Each edge is construed to have two *edge-ends*. Using the arrow convention on the edges in our figures enables us to distinguish the edge-ends. The end at the tail of the arrow on edge j is called j^+, and the end at the head is called j^-.

It is presumed that the reader is fully familiar (e.g., see [GrTu87]) with the representation of the *rotation at a vertex* v as a cyclic ordering of the edge-ends incident at v. This convention enables us to represent a walk as a sequence of edge-ends, which are the edge-ends at which the respective oriented edges in the trail terminate. We sometimes also write the vertices encountered along the walk. We refer to a face-boundary walk of a graph imbedding as an ***fb-walk***. It is further presumed that the reader is familiar with the construction of a graph imbedding from a *rotation system*, by which we mean the assignment of a rotation at every vertex.

Section 2 introduces the concepts of the *local genus polynomial* and *local log-concavity*. We formulate Conjecture 2.2, a topological conjecture, called the *Local Log-Concavity Conjecture*. This conjecture has a purely combinatorial equivalent form, given in Section 3 as Conjecture 3.1. Section 4 gives an elementary proof of local log-concavity for vertices of degree four or less. Section 5 examines how to use Jackson's formula [Ja87] in explorations of the log-concavity of local genus distributions. Section 6 shows how the concept of *synchronicity*, introduced previously by the present authors [GMTW15], provides a plausible path to a predominantly combinatorial proof of the LCGD Conjecture.

This study evolved partially from the general idea of seeking a simplifying effect when one vertex of a graph is removed or added.

2. Local Log-Concavity

A ***partial rotation at a vertex*** v is a cyclic sequence, each of whose elements is either

(1) an edge-end that is incident at v, or
(2) an ellipsis $(\cdots)$.

Although there may be more than one ellipsis, none of the edge-ends may occur more than once. A partial rotation whose only element is an ellipsis is said to be ***empty***. A partial rotation with no ellipsis is said to be ***full***.

A ***partial rotation system*** for a graph is an assignment of a partial rotation to every vertex. We observe that a partial rotation at a vertex v can be extended to a full rotation at v by filling the edge-ends not already used into the gaps indicated by the ellipses. Moreover, a partial rotation system can be extended to a complete rotation system by extending every partial rotation (including any empty partial rotations) to a full rotation.

An imbedding $h : G \to S_j$ of a graph G into a surface is said to be ***consistent with a partial rotation system*** σ for G if the complete rotation system induced by the imbedding h is an extension of σ, in the sense that each row of that complete rotation system is obtainable from the corresponding row of σ by substituting a sequence of missing edge-ends incident at the corresponding vertex for an ellipsis.

The ***partial genus distribution*** of a graph G ***with respect to a partial rotation system*** σ is an assignment to each non-negative integer i of the number $g_{\sigma,i}(G)$ (or $g_{\sigma,i}$) of imbeddings $G \to S_i$ that are consistent with σ. A partial genus distribution is commonly represented by a sequence

$$g_{\sigma,0},\ g_{\sigma,1},\ g_{\sigma,2},\ \cdots$$

or by a polynomial $g_{\sigma,0} + g_{\sigma,1}z + g_{\sigma,2}z^2 + \dots$.

The ***minimum genus (maximum genus) with respect to a partial rotation system*** σ is the least (greatest) integer i such that there exists an imbedding of G consistent with σ in the surface S_i; we use the notations $\gamma_{\sigma,\min}(G)$ and $\gamma_{\sigma,\max}(G)$, respectively. Thus, we may now define the partial genus distribution polynomial by

$$\Gamma_{G,\sigma}(z) = \sum_{i=\gamma_{\sigma,\min}}^{\gamma_{\sigma,\max}} g_{\sigma,i} z^i. \tag{2.1}$$

The partial genus distribution with respect to the empty partial rotation system is called the ***genus distribution*** of G.

We recall that a finite sequence of real numbers $a_0, a_1, \dots, a_n$, with no internal zeros, is ***log-concave*** if for $i = 1, 2, \dots, n-1$, we have

$$a_{i-1}a_{i+1} \le a_i^2.$$

We observe that a sequence is log-concave if and only if the reverse sequence is log-concave.

The following conjecture was first formulated in [GRT89]:

Conjecture 2.1 (***LCGD Conjecture*****).** The genus distribution of every graph is log-concave.

We employ these notations:

$\mathcal{R}_G$**:** the set of all rotation systems on the graph G.

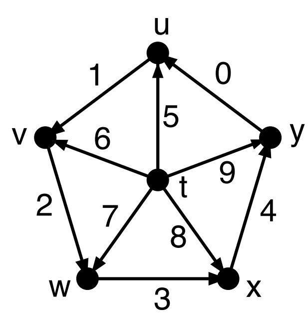
$\mathcal{R}_S$**:** the set of partial rotation systems that specify a full rotation at every vertex of any subset $S \subset V_G$.

The genus distribution of a graph G is said to be ***locally log-concave at a vertex*** v if for every partial rotation system $\sigma \in \mathcal{R}_{V_G - \{v\}}$, the partial genus distribution with respect to the set of (complete) rotation systems for G that are consistent with σ is log-concave.

Proposition 2.1. *Let v be an n-valent vertex of a graph G, and let $\sigma \in \mathcal{R}_{V_G - \{v\}}$. Then the number of complete rotation systems for G that are consistent with the partial rotation system σ is $(n-1)!$.*

Proof. For each possible rotation at v, i.e., for each cyclic ordering of the n edge-ends incident at v, there is exactly one complete rotation system for G that is consistent with σ. □

Example 2.1. We consider the 5-wheel W_5, which is illustrated in Figure 2.1, and a partial rotation system σ that assigns a fixed rotation at every vertex except the hub vertex t. In the figure, the rotation at vertex t is $(5^+6^+7^+8^+9^+)$.



partial rot. sys. σ
u. $0^- \; 1^+ \; 5^-$
v. $1^- \; 2^+ \; 6^-$
w. $2^- \; 3^+ \; 7^-$
x. $3^- \; 4^+ \; 8^-$
y. $0^+ \; 9^- \; 4^-$

FIGURE 2.1. The wheel W_5.

For each of the 24 possible rotations at the 5-valent vertex t, we use face-tracing to calculate the number of fb-walks. Then we use the Euler polyhedral equation to calculate the genus:

(0) For only one of these rotations, namely $(5^+6^+7^+8^+9^+)$ — as in Figure 2.1, the genus of the imbedding surface is 0.
(1) For exactly 15 other rotations, the genus of the imbedding surface is 1.
(2) Each of the remaining 8 rotations at vertex t yields genus 2.

Thus, with respect to the partial rotation system σ, the partial genus distribution of W_5 at vertex t is

$$1 \quad 15 \quad 8$$

which is a log-concave sequence. We may represent that sequence by the polynomial

$$\Gamma_{(W_5,t),\sigma}(z) \;=\; 1 + 15z + 8z^2, \tag{2.2}$$

which we call the ***local genus polynomial with respect to*** σ. In accordance with notation given just before Equation (2.1)

$$\Gamma_{(G,t),\sigma}(z) \;=\; \sum_{i=\gamma_{\sigma,\min}}^{\gamma_{\sigma,\max}} g_{\sigma,i} z^i$$

we have $g_{\sigma,0} = 1$, $g_{\sigma,1} = 15$, and $g_{\sigma,2} = 8$.

To prove local log-concavity of the 5-wheel W_5 at the vertex t, we would need to prove additionally that, with respect to all $31 = 2^5 - 1$ other possible partial rotation systems in the set $\mathcal{R}_{V_{W_5}-\{t\}}$, the partial genus distribution is log-concave.

Comment 2.1. We observe that the genus polynomial and the local genus polynomial of a graph are invariant under subdivision. To avoid cumbersome notation, we can trisect self-loops wherever doing so allows a proof for simple graphs or multi-graphs to be extended to all graphs, i.e., including those with self-loops.

Theorem 2.2. *Let v be a vertex of a graph G. Then the sum of the local genus polynomials at vertex v equals the genus polynomial of G.*

Proof. The set of all imbeddings of G is partitioned into the subsets of imbeddings that are consistent with each possible partial rotation system $\sigma \in \mathcal{R}_{V_G-\{v\}}$. It follows that

$$\Gamma_G(z) \;=\; \sum_{\sigma\in\mathcal{R}_{V_{W_5}-\{v\}}} \Gamma_{(G,v),\sigma}(z) \qquad \square$$

We now formulate the following new conjecture:

Conjecture 2.2 (***Local Log-Concavity Conjecture***, abbreviated as the ***LLC Conjecture*****).** The genus distribution of any graph G is locally log-concave at every vertex of G.

Inner and Outer Strands

Relative to a given vertex v and to a given rotation system ρ, or to the corresponding imbedding, we call the faces and fb-walks that are incident at v ***inner faces*** and ***inner fb-walks***, and we call the other faces and fb-walks ***outer faces*** and ***outer fb-walks***. For instance, the imbedding of the wheel W_5 in Figure 2.1 has five inner faces, all 3-sided, and one outer face, which is 5-sided.

A ***semi-trail*** in a graph is an oriented walk in which no oriented edge appears more than once.

For any imbedding of a graph G, each inner fb-walk at a vertex v can be visualized as having been formed by the union of two kinds of oriented semi-trails:

- a ***corner strand at*** v is a semi-trail (of length 2) consisting of an oriented edge e_{in} ending at v and the oriented edge e_{out} that follows e_{in} immediately after the vertex v on whatever inner fb-walk contains e_{in};
- an ***outer strand at*** v is any semi-trail of an inner fb-walk that remains after deleting all the corner strands from all the fb-walks.

Example 2.2. Figure 2.2 illustrates the configuration of inner and outer strands induced at vertex $u = 0$ of some graph by some imbedding.

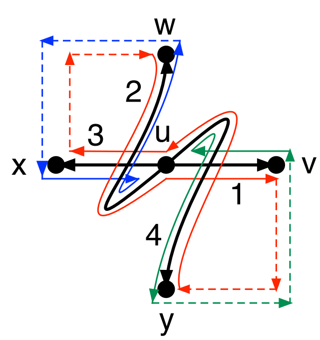


Figure 2.2. Local configuration of corner strands and outer strands.

We observe that there are three fb-walks incident at vertex 0, one in red, another in blue, and another in green. The corner strands

$$v1^-u4^+y \quad w2^-u1^+v \quad x3^-u2^+w \quad y4^-u3^+x \tag{2.3}$$

are represented by solid lines, while the outer strands

$$v \bullet y \quad w \bullet x \quad x \bullet w \quad y \bullet v \tag{2.4}$$

are represented by dashed lines. A bullet (•) stands for an unspecified sequence of edges and vertices lying between the two designated extreme vertices of an outer strand.

We observe the following in Figure 2.2:

(1) The green fb-walk is the union of the corner strand $v1^-u4^+y$ and the outer strand $y \bullet v$.
(2) The blue fb-walk is the union of the corner strand $x3^-u2^+w$ and the outer strand $w \bullet x$.
(3) The red fb-walk is the union of the corner strands $w2^-u1^+v$ and $y4^-u3^+x$ and the outer strands $v \bullet y$ and $x \bullet w$.

Whereas the set of corner strands at a vertex u is completely determined by the rotation at u, the set of outer strands at u is determined by the rotations at all the vertices except for u. Importantly, we observe that there is a bijective correspondence between the set of possible sets of corner strands and the set of rotations at u. At an n-valent vertex u, there are $(n-1)!$ inner permutations.

For any fixed partial rotation system $\sigma \in \mathcal{R}_{V_G - \{u\}}$, all the outer fb-walks are the same, regardless of the rotation at u. It is clear that the number of inner fb-walks incident at an n-valent vertex u is at least 1 and at most n. It follows, from consideration of the Euler polyhedral equation, that the different possible numbers of fb-walks incident at vertex u in these $(n-1)!$ imbeddings all have the same parity. Thus, these $(n-1)!$ imbeddings are distributed over at most $\lfloor \frac{n-1}{2} \rfloor$ values of the genus.

Inner and Outer Permutations

The set of corner strands at a vertex u of an imbedded graph can be represented by what is called the ***inner permutation***, in which the undirected edge at the tail of each corner strand is regarded as being permuted to the undirected edge incident at the head of that strand. In Example 2.2, the inner permutation is $\zeta = (1432)$. It follows immediately from the definition of an inner permutation that the inner permutation for a vertex u of a given imbedding corresponds to the rotation at u.

Similarly, the set of outer strands can be represented by the ***outer permutation***, in which the edge at the tail of each outer strand is permuted to the edge at the head of that strand. In Example 2.2, the outer permutation is $\pi = (14)(23)$. At an n-valent vertex, there may be as many as $n!$ possible outer permutations. We observe that an outer permutation at a vertex u might conceivably be any permutation whatever of the edges incident at u. A given graph might not realize all these possibilities. However, given any permutation $\pi \in \Sigma_n$, it is easy enough to draw an imbedding in which some vertex has π as an outer permutation.

Example 2.3. The rooted wheel graph $(W_4, \circ)$ appears in Figure 2.3. Since the root-vertex is 4-valent, the set $\mathcal{I}$ of ***inner permutations*** has cardinality $(4-1)! = 6$. We observe that

$$(2.5)\quad \mathcal{I} \;=\; \{(0\,1\,2\,3),\,(0\,1\,3\,2),\,(0\,2\,1\,3),\,(0\,2\,3\,1),\,(0\,3\,1\,2),\,(0\,3\,2\,1)\}$$

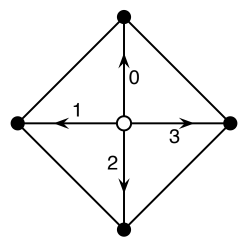


FIGURE 2.3. The rooted wheel graph $(W_4, \circ)$.

Since there are four other vertices, each 3-valent, the set $\mathcal{O}$ of outer permutations has cardinality $((3-1)!)^4 = 16$. Using symmetries, the five rotation projection diagrams in Figure 2.4 are sufficient to derive the multi-set of outer permutations.

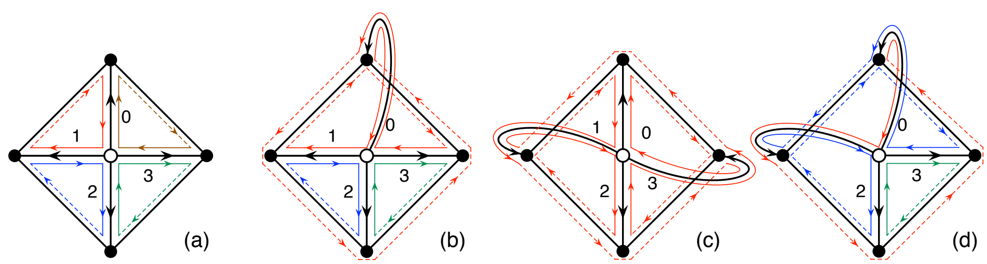


FIGURE 2.4. Outer permutations of the wheel graph W_4.

Since the value of a local genus polynomial $\Gamma_{(G,v),\sigma}(z)$ depends only on the outer permutation π at the root-vertex v of a graph G, rather than on the entire partial rotation system σ, we can sometimes, depending on context, denote the local genus polynomial by $\Gamma_{(G,v),\pi}(z)$.

The outer permutation corresponding to Figure 2.4(a) is $\pi = (0\,3\,2\,1)$. Using the Euler polyhedral equation, we easily calculate that the local genus polynomial is $\Gamma_{(W_4,\circ),\pi}(z) = 1+5z$. When the rotations at all the vertices on the rim of the wheel are reversed, the outer permutation is (0 1 2 3), and the same local genus polynomial $\Gamma_{(W_4,\circ),\pi}(z) = 1 + 5z$ is obtained.

The outer permutation corresponding to the rotation system projection in Figure 2.4(b) is $\pi = (0)(1\ 3\ 2)$. It should be clear that each of the eight outer permutations with one 3-cycle and one 1-cycle leads to the same local genus polynomial $\Gamma_{(W_4,\circ),\pi}(z) = 3z + 3z^2$.

Figure 2.4(c) corresponds to the outer permutation $\pi = (0\ 2)(1\ 3)$. It leads to the local genus polynomial $\Gamma_{(W_4,\circ),\pi}(z) = 4z + 2z^2$. If the rotations at all four rim vertices are reversed, relative to Figure 2.4(c), then the same outer permutation $\pi = (0\ 2)(1\ 3)$ is obtained.

In the imbedding depicted by Figure 2.4(d), the outer permutation is $\pi = (0\ 1)(2\ 3)$. The resulting local genus polynomial is $\Gamma_{(W_4,\circ),\pi}(z) = 4z + 2z^2$. Reversing rotations at all the rim vertices yields the same outer permutation and, thus, the same local genus polynomial. If the rotations at the rim vertices at the heads of edges 1 and 2 are reversed, relative to Figure 2.4(a), and other two rim vertex rotations are as shown in that rotation projection, then the outer permutation is $\pi = (0\ 3)(1\ 2)$, in which case the local genus polynomial is $\Gamma_{(W_4,\circ),\pi}(z) = 4z + 2z^2$. Moreover, if the rotations at the rim vertices at the heads of edges 0 and 3 are reversed, relative to Figure 2.4(a), and other two rim vertex rotations are as shown in that rotation projection, then the outer permutation is also $\pi = (0\ 3)(1\ 2)$.

Summing the local genus distributions at the root of $(W_4, \circ)$ yields the genus distribution of the wheel W_4.

$$
\begin{array}{cccccccc}
(a) & 2(1+5z) & = & 2 & + & 10z & & \\
(b) & 8(3z+3z^2) & = & & & 24z & + & 24z^2 \\
(c) & 2(4z+2z^2) & = & & & 8z & + & 4z^2 \\
(d) & 4(4z+2z^2) & = & & & 16z & + & 8z^2 \\
\hline
 & \Gamma_{W_4}(z) & = & 2 & + & 58z & + & 36z^2
\end{array}
$$

Outer Permutation and Local Genus Polynomial

In Example 2.3, we see that the outer permutations $\pi = (0\ 2)(1\ 3)$ and $\pi = (0\ 1)(2\ 3)$ for Figures 2.4(c) and 2.4(d), respectively, are conjugates in Σ_4, even though the topology of the two figures differs. We observe that both topological configurations lead to the same local genus polynomial $4z+2z^2$. This phenomenon is generalized in the next section, via Corollary 3.3.

The construction of a graph imbedding *a la* Figure 3.1 enables us to realize an arbitrary permutation as an outer permutation. In particular, we can realize the permutations (12)(3)(4) and (1)(2)(3)(4) as outer permutations. These two outer permutations would have the same *local face-count polynomial*, as defined in Section 3. Accordingly, it would be possible for two different conjugacy classes of outer permutation to correspond to the same local genus polynomial.

3. Combinatorial View of Local Log-Concavity

In this section, we show that the LLC Conjecture 2.2 can be proved by affirming a purely combinatorial conjecture, given here as Conjecture 3.1. It will also be shown, conversely, that the LLC Conjecture 2.2 implies Conjecture 3.1.

Remark. In a written composition of permutations, our convention is that the *leftmost* permutation is taken to be applied first.

Proposition 3.1. *Let ζ and π be the inner and outer permutations at a vertex v of an imbedded graph G. Then each cycle of the composition $\zeta\pi$ is the list of the edges incident at v that occur on an inner fb-walk, that is, on an fb-walk incident at v.*

Proof. In effect, the composition of the permutations ζ and π corresponds to the topological operation of pasting corner strands and outer strands together at their extreme vertices, that is, at the neighbors of the vertex v. □

In Example 2.2, we see that $\zeta\pi = (1432) \cdot (14)(23) = (1)(24)(3)$. In Figure 2.2, we observe:

- Edge 1 is the only edge at the tail vertex of a corner strand in the green fb-walk.
- Edge 3 is the only edge at the tail vertex of a corner strand in the blue fb-walk.
- Edges 2 and 4 are incident at the tail vertices of the corner strands of the red fb-walk.

Let Σ_n be the symmetric group on n objects. For $\alpha \in \Sigma_n$, let $c(\alpha)$ denote the number of cycles of α, and let

$$c'(\alpha) \;=\; \left\lfloor \frac{c(\alpha) - 1}{2} \right\rfloor .$$

Of course, the effect of the transformation $c(\pi) \to c'(\pi)$ is to remove the zeros from the sequence of values of $c(\zeta\pi)$ as ζ ranges through the inner permutations. We define

$$Q_n \;=\; \{\zeta \in \Sigma_n \mid c(\zeta) = 1\}$$

to be the set of permutations in Σ_n having exactly one cycle. Moreover, for any permutation $\pi \in \Sigma_n$, regarding π as a possible outer permutation, we define the ***local face-count polynomial***

$$F_\pi(y) \;=\; \sum_{\zeta \in Q_n} y^{c'(\zeta\pi)}. \tag{3.1}$$

By defining f'_i to be the coefficient of y^i in the local face-count polynomial, and by letting d be the degree of the polynomial (3.1), we can write the local face-count polynomial (3.1) in the alternative form

$$F_\pi(y) \;=\; \sum_{i=0}^{d} f'_i\, y^i. \tag{3.2}$$

When π is the outer permutation of a graph (G,t), we can use the form

$$F_{(G,t),\pi}(y) \;=\; \sum_{i=0}^{d} f'_i\, y^i. \tag{3.3}$$

Returning to the wheel graph W_5 of Example 2.1, we have

$$F_{(W_5,t),(12345)}(y) \;=\; 8 + 15y + y^2. \tag{3.4}$$

We recall from Equation (2.2) that

$$\Gamma_{(W_5,t),(12345)}(z) \;=\; 1 + 15z + 8z^2. \tag{3.5}$$

We will now examine the general rule under which a local genus polynomial corresponds to a reversed and transposed local face-count polynomial.

By the ***reverse of the local face-count polynomial*** (3.3), we mean the polynomial

$$F^{\leftarrow}_{(G,t),\pi}(z) \;=\; \sum_{i=0}^{d} f'_{d-i}\, z^i. \tag{3.6}$$

In Example 2.1 regarding the 5-wheel, the graph is planar, and we observe that

$$\Gamma^{\leftarrow}_{(W_5,t),(12345)}(z) \;=\; F_{(W_5,t),(12345)}(z).$$

For a non-planar graph G, there is a right shift of the coefficients of $F^{\leftarrow}_{(W_5,t),(12345)}(z)$ by the number of indices equal to the minimum genus of G. That is,

$$\Gamma_{(G,t),\sigma}(z) \;=\; z^{\gamma_{\min}(G)} F^{\leftarrow}_{(G,t),\pi}(z). \tag{3.7}$$

Invariance of $F_\pi(y)$ over Conjugacy Class

Let $\lambda = (\lambda_1, \ldots, \lambda_s)$ be a partition of n, where $\lambda_1 \geq \ldots \geq \lambda_s$. We say that a permutation $\pi \in \Sigma_n$ is of ***type*** λ if $\{\lambda_1, \ldots, \lambda_s\}$ is the multi-set of the lengths of cycles of π, in which case we may write

$$\tau(\pi) \;=\; \lambda \;=\; (\lambda_1, \ldots, \lambda_s).$$

We now introduce the notation T_π to represent the multi-set of types of $\zeta\pi$ as ζ ranges over all inner permutations. That is,

$$T_\pi \;=\; \{\tau(\zeta\pi) \mid \zeta \in Q_n\}.$$

In Example 2.2, where $\pi = (14)(23)$, we can hand-calculate that

$$T_\pi \;=\; \{(2,1,1), (2,1,1), (2,1,1), (2,1,1), (4), (4)\}. \tag{3.8}$$

This corresponds to the fact that, given the fixed partial rotation system represented by the outer permutation π,

- there are four rotations at vertex v such that, in the resulting complete rotation system for the graph G, one fb-walk is twice incident at v and two fb-walks are once incident at v;
- there are two rotations at vertex v such that, in the resulting complete rotation system for the graph G, a single fb-walk in incident four times at v.

Theorem 3.2. *Let $\pi, \sigma \in \Sigma_n$, with $\tau(\pi) = \tau(\sigma)$. Then $T_\pi = T_\sigma$.*

Proof. It is easy enough to verify that the effect of conjugating the outer permutation π by the permutation α amounts to permuting the symbols of π, while preserving the type. Accordingly, since the inner permutation ζ ranges over all possible cyclic orderings of $[n]$, we have $T_\pi = T_\sigma$. □

Corollary 3.3. *Let $\pi, \sigma \in \Sigma_n$, with $\tau(\pi) = \tau(\sigma)$. Then $F_\pi(y) = F_\sigma(y)$.*

Proof. Clearly, the polynomial $F_\pi(y)$ depends only on the multi-set T_π. For instance, from (3.8), we easily calculate for Example 2.2 that $F_{(14)(23)}(y) = 2 + 4y$. □

The Combinatorial Local Log-Concavity Conjecture

Conjecture 3.1 (***Combinatorial Local LC Conjecture***, sometimes abbreviated as the ***CLLC Conjecture***)**.** Let $\pi \in \Sigma_n$, for any $n \geq 1$. Then the polynomial $F_\pi(y)$ is log-concave.

In experimental computer calculations, the authors have verified Conjecture 3.1 for all $n \leq 12$.

Theorem 3.4. *For a permutation $\pi \in \Sigma_n$, let the local face-count polynomial $F_\pi(y)$ be log-concave. Then the partial genus distribution for a partial rotation system $\sigma \in \mathcal{R}_{V_G - \{v\}}$, for a graph G, with outer permutation π is log-concave.*

Proof. Of course, the set of outer fb-walks is fixed. The inner permutation ζ varies over all cyclic permutations of the neighbors of v, and the number of fb-walks incident at v corresponds to the number of cycles in the composition of the inner and outer permutations.

The number of faces of an imbedding equals the sum of the number of outer fb-walks and the number of inner fb-walks. Since the number of vertices and the number of faces are constant, it follows from the Euler polyhedral equation that the partial genus distribution taken over all imbeddings that are consistent with the partial rotation system σ corresponds to reversal as per Equation (3.6) and a translation as per Equation (3.7) of the local face-count polynomial. Since the local face-count polynomial $F_\pi(y)$ is log-concave, it now follows that the local genus distribution polynomial corresponding to σ is log-concave. $\square$

Corollary 3.5. *If the CLLC Conjecture 3.1 is true, then so is the LLC Conjecture 2.2.* $\square$

Equivalence of the LLC and CLLC Conjectures.

To prove that the LLC Conjecture 2.2 and the CLLC Conjecture 3.1 are equivalent, it is sufficient, in consideration of Corollary 3.5, to prove that Conjecture 2.2 implies Conjecture 3.1.

Theorem 3.6. *The LLC Conjecture 2.2 implies the CLLC Conjecture 3.1.*

Proof. Assume that the LLC Conjecture 2.2 is true. Let $\pi \in \Sigma_n$. We describe how to construct a graph G with a vertex 0 and an imbedding $G \to S_0$ such that a permutation of the same type as π is the outer permutation for v. In view of Corollary 3.3, that is sufficient.

We shall suppose that π has k cycles, of lengths $c_1, c_2, \ldots, c_k$. For $j = 1, \ldots, k$, let $s_j = \sum_{i=1}^{j} c_j$. Thus, the permutation π is of the form

$$\pi \;=\; (1, 2, \ldots, s_1)(s_1{+}1, s_1{+}2, \ldots, s_2) \cdots (s_{k-1}{+}1, s_{k-1}{+}2, \ldots, s_k).$$

We begin the construction of the graph imbedding $G \to S_0$ by drawing a sequence of k cycle graphs in the sphere S_0 so that none of the cycle graphs separates two other cycle graphs from each other. The j^{th} cycle graph is a c_j-cycle, whose arcs are labeled $s_{j-1}+1$, $s_{j-1}{+}2$, $\ldots$, s_j, consecutively in clockwise order around the cycle. To complete the imbedding $G \to S_0$, we then place a vertex v into the region of which every one of the cycles is a boundary component, and we join the vertex v to each vertex on every cycle. Such a graph imbedding is illustrated in Figure 3.1, for the permutation $\pi = (123)(45)(6)(789)$. Let σ be the partial rotation system for G comprising the rotation at every vertex except v, for the imbedding $G \to S_0$.

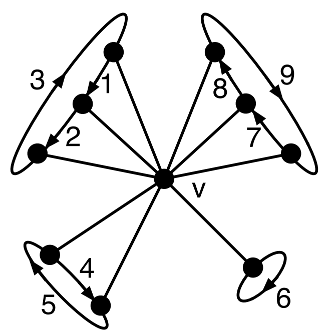


FIGURE 3.1. Realizing (123)(45)(6)(789) as an outer permutation.

Taking the LLC Conjecture 2.2 to be true, it follows that the partial genus distribution with respect to the partial rotation system σ is log-concave. Of course, the corresponding partial genus distribution polynomial is a reversal and translation of the local face-count polynomial $F_\pi(y)$. $\square$

REAL-ROOTEDNESS

Stahl [Stah97] conjectured that the genus polynomial was real-rooted. However, Chen and Liu [ChLi10] published a counterexample. We explored the local face-count polynomials $F_\pi(y)$ for $\pi \in \Sigma_n$ with $n \leq 12$, and we discovered that they are all real-rooted. Nonetheless, in view of [ChLi10], we see insufficient cause to expect the local face-polynomials to be real-rooted.

4. Permutations for Which $F_\pi(y)$ is Log-Concave

In this section, we identify some conjugacy classes of permutations $\pi \in \Sigma_n$ for which the local face-count polynomial $F_\pi(y)$ is log-concave. To simplify the notations involved in discussing the compositions $\zeta\pi$, we take $(1\ 2\ \ldots\ n)$ as the generic n-cycle, rather than $(i_1\ i_2\ \ldots\ i_n)$, and we allow ζ to range over all n-cycles in Σ_n. Since our context is the pursuit of information about graph imbeddings, we continue to refer to $F_\pi(y)$ as a local face-count polynomial, even though it is calculable purely in the abstract.

We recall that the ***bouquet*** B_n is the graph with one vertex and n self-loops. In the course of proving that the genus distributions of bouquets are log-concave, Gross, Robbins, and Tucker [GRT89] used a formula of Jackson [Ja87], upon which we elaborate in Section 5, to prove the following:

Theorem 4.1 (Theorem 4.1 of [GRT89])**.** *Let π be a full involution, that is, with no fixed points. Then $F_\pi(y)$ is log-concave. In other words, the CLLC Conjecture holds for a full involution.* □

To apply Theorem 4.1 to the bouquet B_n, we trisect each of the n self-loops. That is, we subdivide it into three edges. Then the outer permutation π is the involution that interchanges the two 2-valent vertices on each subdivided self-loop.

We now proceed to prove that several additional types of permutations beyond full involutions have log-concave local face-count polynomials.

Theorem 4.2. *For any of the following four types of permutations $\pi \in \Sigma_n$, the local face-count polynomial $F_\pi(y)$ is log-concave.*

(a). a transposition (along with some 1-cycles);
(b). a 3-cycle (along with some 1-cycles);
(c). the composition of two transpositions (along with some 1-cycles);
(d). a 4-cycle (along with some 1-cycles).

Proof of (a). Suppose that $\pi = (i\ j)$. We observe that

$$(1\ 2\ \ldots\ n)\cdot(i\ j) \;=\; (1\ \ldots\ i{-}1\ j\ j{+}1\ \ldots\ n)(i\ i{+}1\ \ldots\ j{-}1).$$

Similarly, for any full cycle ζ, the composition $\zeta\pi$ has two cycles. It follows that the local face-count polynomial $F_\pi(y)$ is the constant polynomial $(n-1)!$, which is log-concave.

Proof of (b). Let $\pi = (i\ j\ k)$, where $i < j$ and $i < k$. We standardize the representation of n-cycles, so that the first symbol is always 1. In half of those n-cycles, the symbol j precedes k, and we have

$$(1\,2\,\ldots\,n)\cdot(i\,j\,k) \;=\; (1\,\ldots\,i{-}1\;j\;j{+}1\,\ldots\,k{-}1\;i\;i{+}1\,\ldots\,j{-}1\;k\;k{+}1\,\ldots\,n).$$

In the other half of the n-cycles, the symbol k precedes j, and we have

$$\begin{aligned}(1\,2\,\ldots\,n)\cdot(i\,j\,k) \;&=\; (1\;\ldots\;i{-}1\;j\;j{+}1\;\ldots\;n)\\ &\quad\cdot(i\;i{+}1\;\ldots\;k{-}1)(k\;k{+}1\;\ldots\;j{-}1).\end{aligned}$$

It follows that $F_\pi(y) = ((n{-}1)!/2){+}((n{-}1)!/2)y$, which is log-concave.

Proof of (c). Let $\pi = (i\ j)(k\ l) \in \Sigma_n$ be the product of two transpositions. In view of Parts (a) and (b), we may assume that i, j, k, l are distinct. We assume also that i is the least of the four. By symmetry in k and l, we can group six cases into three pairs.

Case 1: $j < k < l$ (and, equivalently, $j < l < k$). We have

$$\begin{aligned}(1\,2\,\ldots\,n)\cdot(i\,j)(k\,l) \;&=\; (1\;\ldots\;i{-}1\;j\;j{+}1\,\ldots\,k{-}1\;l\,\ldots\,n)\\ &\quad\cdot(i\;i{+}1\;\ldots\;j{-}1)(k\;k{+}1\ldots\,l{-}1).\end{aligned}$$

Case 2: $k < j < l$ (and, equivalently, $l < j < k$). We have

$$\begin{aligned}&(1\,2\,\ldots\,n)\cdot(i\,j)(k\,l) \;=\\ &\qquad (1\;\ldots\;i{-}1\;j\;j{+}1\,\ldots\,l{-}1\;k\,\ldots\,j{-}1\;i\;i{+}1\;\ldots\;k{-}1\;l\;l{+}1\ldots\,n)\,.\end{aligned}$$

Case 3: $k < l < j$ (and, equivalently, $l < k < j$). We have

$$\begin{aligned}(1\,2\,\ldots\,n)\cdot(i\,j)(k\,l) \;&=\; (1\,\ldots\,i{-}1\;j\;j{+}1\,\ldots\,n)(k\;k{+}1\,\ldots\,l{-}1)\\ &\quad\cdot(i\,i{+}1\,\ldots\,k{-}1\;l\;l{+}1\ldots\,j{-}1).\end{aligned}$$

It follows that $F_\pi(y) = \dfrac{(n-1)!}{3} + \dfrac{2(n-1)!}{3}y$, which is log-concave.

Proof of (d). Let $\pi = (i\ j\ k\ l) \in \Sigma_n$ be a 4-cycle. By the same form of analysis as in the proofs of Parts (b) and (c), we calculate that $F_\pi(y) = \dfrac{5(n-1)!}{6} + \dfrac{(n-1)!}{6}y$, which is log-concave. □

Conjecture 4.1. Let $\pi \in \Sigma_n$ be the composition of a q-cycle for $q \le n$ and some 1-cycles. Then $F_\pi(y)$ is log-concave.

Conjecture 4.2. Let $\pi \in \Sigma_n$ be any involution. Then $F_\pi(y)$ is log-concave.

Conjecture 4.3. Let $\pi \in \Sigma_n$ be any permutation such that $F_\pi(y)$ is log-concave, and let τ be a transposition. Then $F_{\pi\tau}(y)$ is log-concave. (This conjecture immediately implies the CLLC Conjecture 3.1, and accordingly, the LLC Conjecture 2.2.)

An Immediate Application to Graphs

Theorem 4.3. *Let v be a vertex of degree four or less in a graph G. Then the genus distribution of G is locally log-concave at v.*

Proof. For any partial rotation system $\sigma \in \mathcal{R}_{V_G - \{v\}}$, let π be the outer permutation. Since the degree of vertex v is four or less, we know that the outer permutation has one of the following types:

$$(1),\ (2),\ (1,1),\ (3),\ (2,1),\ (1,1,1),\ (4),\ (3,1),\ (2,2),\ (2,1,1),\ (1,1,1,1).$$

It follows from Theorem 4.2 that the local face-count polynomial $F_\pi(y)$ is log-concave for any of these types of outer permutation. It then follows from Theorem 3.4 that the genus distribution of graph G at vertex v is locally log-concave. □

5. Transforming Jackson's Formula

In this section, we derive a purely combinatorial theorem that enables a formula of Jackson [Ja87] to be used in proving theorems about the log-concavity of local genus distributions, as well as in solving other problems on genus distributions.

Let k and N be non-negative integers, let ψ be a partition of N, and let C^ψ be the conjugacy class of ψ. Let ζ_0 be an arbitrary (but fixed) N-cycle in the symmetric group Σ_N. For definiteness, we take

$$\zeta_0 \;=\; \begin{pmatrix}1 & 2 & \cdots & N\end{pmatrix}.$$

We define T_k^ψ to be the subset of Σ_N such that $\pi \in T_k^\psi$ if and only if

(1) $(\exists \xi \in C^\psi)$ such that $\pi = \zeta_0 \xi$;
(2) π has exactly k cycles (including 1-cycles).

Jackson defines

$$e_k^\psi \;=\; \left| T_k^\psi \right|.$$

Corollary 2.5 of [Ja87] expresses $e_k^\psi(N)$ as a character sum. It is indicated by [GRT89] how the sequence

$$e_1^\psi,\ e_2^\psi,\ \ldots,\ e_N^\psi \tag{5.1}$$

is related to problems regarding the genus distributions of the graphs B_n known as *bouquets*. We are seeking to generalize the results there regarding the relationship between the sequence (5.1) and genus distributions.

Our immediate concern is the special case in which the integer N is partitioned into n parts of equal size p. We define $C^{(p)}(n)$ to be the conjugacy class of permutations in Σ_{np} with n p-cycles. We observe that $C^{(N)}(1)$ is the conjugacy class of N-cycles in Σ_N.

We define $T_k^{(p)}(n)$ to be the subset of Σ_{np} such that $\pi \in T_k^{(p)}(n)$ if and only if

(1) $\exists \xi \in C^{(p)}(n)$ such that $\pi = \zeta_0 \xi$;
(2) π has exactly k cycles (including 1-cycles).

That is,

$$T_k^{(p)}(n) \;=\; \{\pi = \zeta_0 \xi \mid \xi \in C^{(p)}(n)\} \tag{5.2}$$

Jackson defines

$$e_k^{(p)}(n) \;=\; \left| T_k^{(p)}(n) \right| \tag{5.3}$$

Theorem 5.4 of [Ja87] gives an explicit expression for $e_k^{(p)}(n)$ in terms of Stirling numbers. Section 6 of [Ja87] shows that $e_k^{(2)}(n)$ and $e_k^{(3)}(n)$ satisfy linear recurrences with polynomial coefficients.

Example 5.1. To calculate $e_k^{(2)}(2)$ for $k = 1, 2, 3$, we start by listing

$$C^{(2)}(2) \;=\; \{(12)(34),\, (13)(24),\, (14)(23)\}.$$

We then calculate

$$\begin{aligned} \zeta_0 C^{(2)}(2) \;=\; \{\, & (1234)(12)(34) = (1)(24)(3), \\ & (1234)(13)(24) = (1432), \\ & (1234)(14)(23) = (13)(2)(4)\} \end{aligned}$$

from which it follows that

$$\begin{aligned} T_1^{(2)}(2) &= \{(1432)\} \\ T_2^{(2)}(2) &= \emptyset \\ T_3^{(2)}(2) &= \{(1)(24)(3),\, (13)(2)(4)\}, \end{aligned}$$

and, in turn, that

$$e_1^{(2)}(2) = \left|T_1^{(2)}(2)\right| = 1$$
$$e_2^{(2)}(2) = \left|T_2^{(2)}(2)\right| = 0$$
$$e_3^{(2)}(2) = \left|T_2^{(2)}(2)\right| = 2.$$

Another Enumeration Problem

As earlier in this section, let k and N be non-negative integers, and let ψ be a partition of N. Rather than fixing a cycle, here we let ξ_0 be an arbitrary (but fixed) permutation in the conjugacy class C^ψ. We define $U_k^{(p)}(n)$ to be the subset of Σ_{np} such that $\pi \in U_k^{(p)}(n)$ if and only if

(1) $\exists \zeta \in C^{(np)}(1)$ such that $\pi = \zeta\xi_0$;
(2) π has exactly k cycles (including 1-cycles).

That is,

$$U_k^{(p)}(n) \;=\; \{\pi = \zeta\xi_0 \mid \zeta \in C^{(np)}(1)\}. \tag{5.4}$$

Analogous to Jackson's $e_k^{(p)}(n)$, we define

$$f_k^{(p)}(n) \;=\; \left|U_k^{(p)}(n)\right|. \tag{5.5}$$

Example 5.1 (continued). Here we take $\xi_0 = (12)(34)$, and we now hand-calculate that for the conjugacy class of $\psi = 22$, we have

$$C^{(4)}(1) \;=\; \{(1234), (1243), (1324), (1342), (1423), (1432)\}.$$

We then calculate

$$\begin{aligned} C^{(4)}(1)\xi_0 \;=\; \{\, & (1234)(12)(34) = (1)(24)(3), \\ & (1243)(12)(34) = (1)(23)(4), \\ & (1324)(12)(34) = (1423), \\ & (1342)(12)(34) = (14)(2)(3), \\ & (1423)(12)(34) = (1324), \\ & (1432)(12)(34) = (13)(2)(4) \,\} \end{aligned}$$

from which it follows that

$$\begin{aligned} U_1^{(2)}(2) &= \{(1324), (1432)\} \\ U_2^{(2)}(2) &= \emptyset \\ U_3^{(2)}(2) &= \{(1)(24)(3),\, (1)(23)(4), (14)(2)(3), (13)(2)(4)\} \end{aligned}$$

and that

$$\begin{aligned} f_1^{(2)}(2) &= 2 \\ f_2^{(2)}(2) &= 0 \\ f_3^{(2)}(2) &= 4. \end{aligned}$$

We observe the following:

$$\begin{aligned} f_1^{(2)}(2) = 2 = 2e_1^{(2)}(2) = 2 \cdot 1 \\ f_2^{(2)}(2) = 0 = 2e_2^{(2)}(2) = 2 \cdot 0 \\ f_3^{(2)}(2) = 4 = 2e_3^{(2)}(2) = 2 \cdot 2. \end{aligned}$$

That is, for $k = 1, 2, 3$, we have $f_k^{(2)}(2) = 2e_k^{(2)}(2)$.

A General Many-to-One Correspondence

We want to establish a general many-to-one correspondence between the cardinalities of the sets of permutations when ξ_0 is fixed, i.e., the sets $C^{(np)}(1)$, and the cardinalities of the sets of permutations when ζ_0 is fixed, i.e., the sets $C^{(p)}(n)$. Proofs of the following three propositions are elementary.

Proposition 5.1. *The cardinality of $C^{(np)}(1)$ is $(np-1)!$.*

Proposition 5.2. *The cardinality of $C^{(p)}(n)$ is*

$$\binom{np}{p \cdots p} \frac{((p-1)!)^n}{n!} \;=\; \frac{(np)!}{p^n n!}.$$

Proposition 5.3. *We have*

$$\frac{\left|C^{(np)}(1)\right|}{\left|C^{(p)}(n)\right|} \;=\; p^{n-1}(n-1)!.$$

This lemma follows from Proposition 1.3.2 of [Stan12].

Lemma 5.4. *Let $\pi \in \Sigma_n$ be a permutation with cycles of distinct lengths $n_1, n_2, \ldots, n_r$, such that there are m_i cycles of length n_i. Then the order of its centralizer is*

$$|C_N(\pi)| \;=\; \prod_{i=1}^{r} n_i^{m_i} m_i!. \tag{5.6}$$

□

Theorem 5.5 reduces the Local Log-Concavity Conjecture for various conjugacy classes of outer permutations to proving that Sequence (5.1) (with alternating zeros omitted) is log-concave. Proof for $p = 2$ appears in [GRT89].

Theorem 5.5.

$$\frac{f_k^{(p)}(n)}{e_k^{(p)}(n)} \;=\; \frac{\left|U_k^{(np)}(1)\right|}{\left|T_k^{(p)}(n)\right|} \;=\; p^{n-1}(n-1)!.$$

Proof. We consider the equation $\pi = \zeta\xi$, where ζ is from the conjugacy class $C^{(np)}(1)$, where ξ is from the conjugacy class $C^{(p)}(n)$, and where π has k cycles. We let ζ_0 be a fixed np-cycle and ξ_0 a fixed element of $C^{(p)}(n)$. We recall from Equation (5.4) and Equation (5.2) that

$$U_k^{(p)}(n) \;=\; \{\pi = \zeta\xi_0 \mid \zeta \in C^{(np)}(1)\}$$

and that

$$T_k^{(p)}(n) \;=\; \{\pi = \zeta_0\xi \mid \xi \in C^{(p)}(n)\}.$$

We observe here that

$$U_k^{(p)}(n) \;=\; \{\pi = \alpha\zeta_0\alpha^{-1}\xi_0 \mid \alpha \in \Sigma_{np}\}$$

and that the cardinality of $U_k^{(p)}(n)$ equals the index of the centralizer of ζ_0 in Σ_{np}. Similarly,

$$T_k^{(p)}(n) \;=\; \{\pi = \zeta_0\alpha^{-1}\xi_0\alpha \mid \alpha \in \Sigma_{np}\}.$$

and the cardinality of $T_k^{(p)}(n)$ equals the index of the centralizer of ξ_0 in Σ_{np}. We now have

$$\begin{aligned}
\frac{f_k^{(p)}(n)}{e_k^{(p)}(n)} &= \frac{\left|U_k^{(np)}(1)\right|}{\left|T_k^{(p)}(n)\right|} \\
&= \frac{|\Sigma_{np} : C_{np}((1\,2\,3\,4\cdots 2n{-}1\,2n))|}{|\Sigma_{np} : C_{np}((1\,2)(3\,4)\cdots(2n{-}1\,2n))|} \\
\therefore\; \frac{f_k^{(p)}(n)}{e_k^{(p)}(n)} &= \frac{|C_{np}((1\,2)(3\,4)\cdots(2n{-}1\,2n))|}{|C_{np}((1\,2\,3\,4\cdots 2n{-}1\,2n))|}.
\end{aligned} \tag{5.7}$$

Applying Lemma 5.4 to Equation (5.7), we have

$$\frac{f_k^{(p)}(n)}{e_k^{(p)}(n)} \;=\; \frac{p^n n!}{pn} \;=\; p^{n-1}(n-1)!.$$

□

6. Synchronicity

The connection between the genus distribution of a graph and the local genus distributions is not readily apparent. Theorem 2.2 establishes that the genus distribution polynomial of a graph is the sum of the local genus distributions at an arbitrary vertex v with respect to all of the partial rotation systems $\sigma \in \mathcal{R}_{V_G-\{v\}}$. However, the sum of log-concave polynomials need not be log-concave. Conversely, the fact that the sum of a set of polynomials is log-concave does not imply that the summands are log-concave. In this section we show how the concept of *synchronicity* of [GMTW15] might be used toward a proof of the LCGD Conjecture.

We say that two nonnegative sequences A and B are ***synchronized***, denoted as $A \sim B$, if both are log-concave, and they satisfy

$$a_{k-1}b_{k+1} \le a_k b_k \text{ and } a_{k+1}b_{k-1} \le a_k b_k \qquad \text{for all } k.$$

It is clear that the synchronicity relation $\sim$ is symmetric. We should be aware of that it is not transitive. There is an easy direct proof of the following theorem.

Theorem 6.1. *Let the three sequences A, B, C be log-concave, nonnegative, and mutually synchronized. Then $A + B$ is log-concave and $A + B \sim C$.* □

The following conjecture has a topological component, since we don't know what multi-sets of permutations are realizable as a set of outer permutations for a graph.

Conjecture 6.1. For any rooted graph (G, v), the multi-set of local genus polynomials

$$\left\{\Gamma_{G,\pi(\sigma_j)}(z) \mid \sigma_j \in \mathcal{R}_{V_G-\{v\}}\right\}$$

can be linearly ordered so that, for any n, the polynomials

$$\sum_{j=1}^{n-1} \Gamma_{G,\pi(\sigma_j)}(z) \text{ and } \Gamma_{G,\pi(\sigma_n)}(z)$$

are synchronized.

Theorem 6.2. *The Local Log-Concavity Conjecture 2.2 and Conjecture 6.1 together imply the LCGD Conjecture.*

Proof. Let (G, v) be a rooted graph. First assume that Conjecture 2.2 is true. Then all the local genus polynomials at vertex v are log-concave.

Next assume that Conjecture 6.1 is true. Then the multi-set of local genus polynomials

$$\left\{\Gamma_{G,\pi(\sigma_j)}(z) \mid \sigma_j \in \mathcal{R}_{V_G-\{v\}}\right\}$$

can be linearly ordered so that, for any n, the polynomials

$$\sum_{j=1}^{n-1} \Gamma_{G,\pi(\sigma_j)}(z) \text{ and } \Gamma_{G,\pi(\sigma_n)}(z)$$

are synchronized. It follows, in turn, from Theorem 6.1 that the sum of all the local genus polynomials is log-concave. We recall Theorem 2.2, that the sum of the local genus polynomials at vertex v equals the genus polynomial of v. Accordingly, the genus polynomial of the graph G is log-concave. □

7. Possible Further Investigations

The results here suggest various additional avenues of research into local log-concavity.

Planar Families and Sub-Families

Proving log-concavity of all planar graphs has been an elusive target. However, there has been success for a collection of more specific families, starting with ladders [FGS89] and bouquets [GRT89]. Most planar families known to have the LCGD property have bounded degree. For instance, although the genus polynomial of fan graph has been determined [CMZ13], the genus polynomial of wheel graphs remains unknown. Of course, the LCGD question is not necessarily answered by calculation of a genus polynomial, nor is calculation of a closed form needed to prove its log-concavity. Exploratory local LC investigations of some planar families, either with known or unknown genus polynomials, might uncover further significant relationships between LCGD and local LC.

Operations That Preserve Local LC

In general, and with special importance to the LCGD properties of recursively specified families of graphs, it would be quite helpful to know which of the most common graph operations preserve LCGD or local LC. An operation that comes to mind immediately is joining a vertex to a graph G. If the graph G is planar, then the resulting graph is called an ***apex graph*** (e.g., see [Moh01]). A highly specific question is under what conditions on a planar graph with the LCGD property or local LC property does forming an apex graph from it preserve those properties.

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