Learning latent variable models using tensor decompositions

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El tema (subject matter)

Learning algorithms for latent variable models based on decompositions of moment tensors.
Learning algorithms (parameter estimation) for latent variable models based on decompositions of moment tensors.

“Method-of-moments” (Pearson, 1894)
Example #1: summarizing a corpus of documents

Observation: documents express one or more thematic topics.

Politics Ensnare Mohamed Salah and Switzerland at the World Cup

By Rory Smith, James Montague and Tariq Panja

June 24, 2018

MOSCOW — The World Cup was thrust into the combustible mix of politics and soccer — dangerous ground that world soccer takes great pains to avoid — as a growing number of disciplinary proceedings and a star player’s threatened retirement brought several sensitive international flash points to the tournament’s doorstep this weekend.
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► What topics are expressed in a corpus of documents?
► How prevalent is each topic in the corpus?
Topic model (e.g., latent Dirichlet allocation)

$K$ topics \(\text{(distributions over vocab words)}\).

Document $\equiv$ mixture of topics.

Word tokens in doc. iid $\sim$ mixture distribution.
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E.g., $\sim 0.7 \times P_{\text{sports}} + 0.3 \times P_{\text{politics}}$. 
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- Word tokens in doc. $\sim$ mixture distribution.

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Given corpus of documents (and “hyper-parameters”, e.g., $K$), produce estimates of model parameters, e.g.:

- Distribution $P_t$ over vocab words, for each $t \in [K]$.
- Weight $w_t$ of topic $t$ in document corpus, for each $t \in [K]$. 
Suppose each word token $x$ in document is *annotated* with source topic $t_x \in \{1, 2, \ldots, K\}$.

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<thead>
<tr>
<th>Politics</th>
<th>Ensnare</th>
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Then estimating the $\{(P_t, w_t)\}_{t=1}^K$ can be done “directly”.
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Then estimating the \( \{(P_t, w_t)\}_{t=1}^K \) can be done “directly”.

Unfortunately, we often don’t have such annotations (i.e., data are *unlabeled* / topics are *hidden*). “Direct” approach to estimation unavailable.
Example #2: subpopulations in data

Data studied by Pearson (1894):
ratio of forehead-width to body-length for 1000 crabs.
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Data studied by **Pearson (1894)**:
ratio of forehead-width to body-length for 1000 crabs.

Sample may be comprised of different sub-species of crabs.
Gaussian mixture model

\[ H \sim \text{Categorical}(\pi_1, \pi_2, \ldots, \pi_K); \]
\[ X \mid H = t \sim \text{Normal}(\mu_t, \Sigma_t), \quad t \in [K]. \]
Gaussian mixture model

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Estimate mean vector, covariance matrix, and mixing weight of each subpopulation from unlabeled data.
No “direct” estimators when some variables are hidden.
Maximum likelihood estimation

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- **Maximum likelihood estimator (MLE):**

\[
\theta_{\text{MLE}} := \arg \max_{\theta \in \Theta} \log \Pr_\theta \text{(data)}.
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- For latent variable models, often use local optimization, most notably via Expectation-Maximization (EM) (Dempster, Laird, & Rubin, 1977).
MLE for Gaussian mixture models

Given data \( \{x_i\}_{i=1}^n \), find \( \{(\mu_t, \Sigma_t, \pi_t)\}_{t=1}^K \) to maximize

\[
\sum_{i=1}^n \log \left( \sum_{t=1}^K \pi_t \cdot \frac{1}{\det(\Sigma_t)^{1/2}} \exp \left\{ -\frac{1}{2} (x_i - \mu_t)^\top \Sigma_t^{-1} (x_i - \mu_t) \right\} \right).
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- Sensible with restrictions on \( \Sigma_t \) (e.g., \( \Sigma_t \succeq \sigma^2 \mathbf{I} \)).
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- Sensible with restrictions on \( \Sigma_t \) (e.g., \( \Sigma_t \succeq \sigma^2 I \)).
- But \textbf{NP-hard} to maximize (Tosh and Dasgupta, 2018):
  Can’t expect efficient algorithms to work for all data sets.
Parameter learning objective

Suppose iid sample of size $n$ is generated by distribution from model with (unknown) parameters $\theta \in \Theta \subseteq \mathbb{R}^p$. ($p = \#$ params)
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**Task**: Produce estimate $\hat{\theta}$ of $\theta$ such that

$$\mathbb{E} \| \hat{\theta} - \theta \| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty$$

(i.e., $\hat{\theta}$ is consistent).

▶ E.g., for spherical Gaussian mixtures:

For $K = 2$ (and $\pi_t = 1/2$, $\Sigma_t = I$): EM is consistent (Xu, H., & Maleki, 2016).

▶ Larger $K$: easily trapped in local maxima, far from global max (Jin, Zhang, Balakrishnan, Wainwright, & Jordan, 2016).

Practitioners often use EM with many (random) restarts ... but may take a long time to get near the global max.
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$$\Pr \left( \| \hat{\theta} - \theta \| \leq \epsilon \right) \geq 1 - \delta$$

with $\text{poly}(p, 1/\epsilon, 1/\delta, \ldots)$ sample size and running time.

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Barriers

Hard to learn model parameters, even when data is generated by a model distribution.
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Cryptographic hardness
(e.g., Mossel & Roch, 2006)

Information-theoretic hardness
(e.g., Moitra & Valiant, 2010)

May require $2^{\Omega(K)}$ running time or $2^{\Omega(K)}$ sample size.
Ways around the barriers

▶ **Separation conditions.**

E.g., assume mixture component distributions are far apart. 
(Dasgupta, 1999; Arora & Kannan, 2001; Vempala & Wang, 2002; …)
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▶ **Structural assumptions.**

E.g., sparsity, anchor words.
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▶ **Non-degeneracy conditions.**

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This lecture: learning algorithms for non-degenerate instances via method-of-moments.
Method-of-moments at a glance

1. Determine function of model parameters $\theta$ estimatable from observable data:

$\mathbb{E}_{\theta}[f(\mathbf{X})]$ ("moments").

Which moments? Often low-order moments suffice.

2. Form estimates of moments using data (e.g., iid sample):

$\hat{\mathbb{E}}[f(\mathbf{X})]$ ("empirical moments").

3. Approximately solve equations for parameters $\theta$:

$\mathbb{E}_{\theta}[f(\mathbf{X})] = \hat{\mathbb{E}}[f(\mathbf{X})].$

4. ("Fine-tune" estimated parameters with local optimization.)
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Method-of-moments at a glance

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**How?** Algorithms for tensor decomposition.

4. ("Fine-tune" estimated parameters with local optimization.)
A simple example of the method-of-moments

Let $X \sim \text{Normal}(\mu, \sigma^2)$. How to estimate $\sigma^2$ from iid sample?
A simple example of the method-of-moments

Let $X \sim \text{Normal}(\mu, \sigma^2)$. How to estimate $\sigma^2$ from iid sample?

- Consider first- and second-moments: $E[X]$ and $E[X^2]$. 

Formula for $\sigma^2$ in terms of moments:

$$E[X^2] - (E[X])^2 = \sigma^2 + \mu^2 - \mu^2 = \sigma^2.$$ 

Form estimates of $E[X]$ and $E[X^2]$ from iid sample $\{x_i\}_{n=1}^n$:

$$E[X] = \frac{1}{n} \sum_{i=1}^n x_i,$$

$$E[X^2] = \frac{1}{n} \sum_{i=1}^n x_i^2.$$ 

Then estimate $\sigma^2$ with $\hat{\sigma}^2 := E[X^2] - (E[X])^2$.

We'll follow this same basic recipe for much richer models!
A simple example of the method-of-moments

Let $X \sim \text{Normal}(\mu, \sigma^2)$. How to estimate $\sigma^2$ from iid sample?

- Consider first- and second-moments: $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$.
- Formula for $\sigma^2$ in terms of moments:

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  e.g.,

  $$
  \hat{\mathbb{E}}[X] := \frac{1}{n} \sum_{i=1}^n x_i, \quad \hat{\mathbb{E}}[X^2] := \frac{1}{n} \sum_{i=1}^n x_i^2.
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- Then estimate $\sigma^2$ with

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- Form estimates of $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$ from iid sample $\{x_i\}_{i=1}^n$:

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We’ll follow this same basic recipe for much richer models!
Outline

1. Topic model for single-topic documents.
   - Identifiability.
   - Parameter recovery via orthogonal tensor decomposition.

2. Moment decompositions for other models.
   - Mixtures of Gaussians and linear regressions.
   - Multi-view models (e.g., HMMs).
   - Other models (e.g., single-index models).

3. Error analysis.
Other models amenable to moment tensor decomposition

- Models for independent components analysis (Comon, 1994; Frieze, Jerrum, & Kannan, 1996; Arora, Ge, Moitra & Sachdeva, 2012; Anandkumar, Foster, H., Kakade, & Liu, 2012, 2015; Belkin, Rademacher, & Voss, 2013; etc.)


- Mixed-membership stochastic blockmodels (Anandkumar, Ge, H., & Kakade, 2013, 2014)

- Simple probabilistic grammars (H., Kakade, & Liang, 2012)

- Noisy-or networks (Halpern & Sontag, 2013; Jernite, Halpern & Sontag, 2013; Arora, Ge, Ma, & Risteski, 2016)

- Indian buffet process (Tung & Smola, 2014)

- Mixed multinomial logit model (Oh & Shah, 2014)

- Dawid-Skene model (Zhang, Chen, Zhou, & Jordan, 2014)

- Multi-task bandits (Azar, Lazaric, & Brunskill, 2013)

- Partially obs. MDPs (Azizzadenesheli, Lazaric, & Anandkumar, 2016)

...
1. Topic model for single-topic documents
Topic model

General topic model (e.g., Latent Dirichlet Allocation)

$K$ topics \((\text{dists. over words}) \{P_t\}_{t=1}^K\).

Document $\equiv$ mixture of topics (hidden).

Word tokens in doc. \(\text{iid} \sim\) mixture distribution.
Topic model for single-topic documents

$K$ topics (dists. over words) $\{P_t\}_{t=1}^K$. Pick topic $t$ with prob. $w_t$ (hidden).

Word tokens in doc. $\sim P_t$. 

(Answering this question leads to efficient algorithms for estimating parameters!)
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Identifiability

**Generative process:**
Pick $t \sim \text{Categorical}(w_1, w_2, \ldots, w_K)$.
Given $t$, pick $L$ words from $P_t$. 

$\Rightarrow \sum_{t=1}^{K} w_t P_t$. 

Are parameters $\{(P_t, w_t)\}_{t=1}^{K}$ identifiable from single-word documents?
No.
Identifiability

Generative process:
Pick $t \sim \text{Categorical}(w_1, w_2, \ldots, w_K)$.
Given $t$, pick $L$ words from $P_t$.

- $L = 1$: random document (single word) $\sim \sum_{t=1}^{K} w_t P_t$.

Are parameters $\{(P_t, w_t)\}_{t=1}^{K}$ identifiable from single-word documents?
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Identifiability: $L = 2$

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- $L = 2$: 
  
  Random document $\sim \sum_{t=1}^{K} w_t P_t P_t^\top$. Are parameters $\{ (P_t, w_t) \}_{t=1}^{K}$ identifiable from word pairs?
Identifiability: \( L = 2 \)

**Generative process:**

Pick \( t \sim \text{Categorical}(w_1, w_2, \ldots, w_K) \).

Given \( t \), pick \( L \) words from \( P_t \).

\[ L = 2: \]

Regard \( P_t \) as probability vector (\( i \)th entry of \( P_t \) is \( \Pr[\text{word } i] \)).

Joint distribution of word pairs (for topic \( t \)) is given by matrix:

\[
P_t P_t^\top = \begin{bmatrix}
    \text{Pr[words } i, j]\end{bmatrix}
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$$P_t P_t^\top = \begin{bmatrix} \Pr[\text{words } i, j] \end{bmatrix}$$

Random document $\sim \sum_{t=1}^{K} w_t P_t P_t^\top$.

Are parameters $\{(P_t, w_t)\}_{t=1}^{K}$ identifiable from word pairs?
Suppose distribution of word pairs (as a matrix) can be written as

$$M = AA^\top.$$
Simple observation

Suppose distribution of word pairs (as a matrix) can be written as

\[ M = AA^\top. \]

Then it can also be written as

\[ M = (AR)(AR)^\top \]

for any orthogonal matrix \( R \) (because \( R^\top R = I \)).
Identifiability: $L = 2$ counterexample

Parameters $\{(P_1, w_1), (P_2, w_2)\}$ and $\{(^\ast P_1, ^\ast w_1), (^\ast P_2, ^\ast w_2)\}$

$$
(P_1, w_1) = \left(\begin{bmatrix} 0.40 \\ 0.60 \end{bmatrix}, 0.5\right), \quad (P_2, w_2) = \left(\begin{bmatrix} 0.60 \\ 0.40 \end{bmatrix}, 0.5\right);
$$

$$
(^\ast P_1, ^\ast w_1) = \left(\begin{bmatrix} 0.55 \\ 0.45 \end{bmatrix}, 0.8\right), \quad (^\ast P_2, ^\ast w_2) = \left(\begin{bmatrix} 0.30 \\ 0.70 \end{bmatrix}, 0.2\right).
$$

Cannot identify parameters from length-two documents.
Identifiability: \( L = 2 \) counterexample

Parameters \( \{(P_1, w_1), (P_2, w_2)\} \) and \( \{(\tilde{P}_1, \tilde{w}_1), (\tilde{P}_2, \tilde{w}_2)\} \)

\[
(P_1, w_1) = \left( \begin{bmatrix} 0.40 \\ 0.60 \end{bmatrix}, 0.5 \right), \quad (P_2, w_2) = \left( \begin{bmatrix} 0.60 \\ 0.40 \end{bmatrix}, 0.5 \right);
\]

\[
(\tilde{P}_1, \tilde{w}_1) = \left( \begin{bmatrix} 0.55 \\ 0.45 \end{bmatrix}, 0.8 \right), \quad (\tilde{P}_2, \tilde{w}_2) = \left( \begin{bmatrix} 0.30 \\ 0.70 \end{bmatrix}, 0.2 \right);
\]

satisfy

\[
w_1 P_1 P_1^T + w_2 P_2 P_2^T = \tilde{w}_1 \tilde{P}_1 \tilde{P}_1^T + \tilde{w}_2 \tilde{P}_2 \tilde{P}_2^T = \begin{bmatrix} 0.26 & 0.24 \\ 0.24 & 0.26 \end{bmatrix}.
\]
Identifiability: $L = 2$ counterexample

Parameters $\{(P_1, w_1), (P_2, w_2)\}$ and $\{({\tilde{P}}_1, {\tilde{w}}_1), ({\tilde{P}}_2, {\tilde{w}}_2)\}$

$$(P_1, w_1) = \begin{pmatrix} 0.40 \\ 0.60 \end{pmatrix}, \quad (P_2, w_2) = \begin{pmatrix} 0.60 \\ 0.40 \end{pmatrix};$$

$$(\tilde{P}_1, \tilde{w}_1) = \begin{pmatrix} 0.55 \\ 0.45 \end{pmatrix}, \quad (\tilde{P}_2, \tilde{w}_2) = \begin{pmatrix} 0.30 \\ 0.70 \end{pmatrix}$$

satisfy

$$w_1 P_1 P_1^\top + w_2 P_2 P_2^\top = \tilde{w}_1 \tilde{P}_1 \tilde{P}_1^\top + \tilde{w}_2 \tilde{P}_2 \tilde{P}_2^\top = \begin{bmatrix} 0.26 & 0.24 \\ 0.24 & 0.26 \end{bmatrix}. $$

Cannot identify parameters from length-two documents.
Identifiability: \( L = 3 \)

**Documents of length** \( L = 3 \)
Joint distribution of word triple (for topic \( t \)) is given by *tensor*:

\[
P_t \otimes P_t \otimes P_t = \text{Pr}[\text{words } i, j, k]
\]

Random document \( \sim \sum_{t=1}^{K} w_t P_t \otimes P_t \otimes P_t. \)
Identifiability from documents of length three

Claim: If $\{P_t\}_{t=1}^K$ are linearly independent & all $w_t > 0$, then parameters $\{(P_t, w_t)\}_{t=1}^K$ are identifiable from word triples.
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- Claim implied by uniqueness of certain tensor decompositions.
Identifiability from documents of length three

Claim: If $\{P_t\}_{t=1}^K$ are linearly independent & all $w_t > 0$, then parameters $\{(P_t, w_t)\}_{t=1}^K$ are identifiable from word triples.

- Claim implied by uniqueness of certain tensor decompositions.
- Proof is constructive: i.e., comes with an algorithm!
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- Claim implied by uniqueness of certain tensor decompositions.
- Proof is constructive: i.e., comes with an algorithm!

**Next:** Brief overview of tensors.
Tensors of order two

Matrices (tensors of order two): \( M \in \mathbb{R}^{d \times d} \).

- Regard as bi-linear function \( M : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \):

\[
M(ax + x', y) = aM(x, y) + M(x', y);
\]
\[
M(x, ay + y') = aM(x, y) + M(x, y').
\]
Matrices (tensors of order two): \( M \in \mathbb{R}^{d \times d} \).

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  \]

- Can describe \( M \) by \( d^2 \) values \( M(e_i, e_j) =: M_{i,j} \).
  
  \((e_i \text{ is } i\text{th coordinate basis vector.})\)
Matrices (tensors of order two): $M \in \mathbb{R}^{d \times d}$.

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($e_i$ is $i$th coordinate basis vector.)

- Formula using matrix representation:

$$M(x, y) = x^\top My = \sum_{i,j} M_{i,j} \cdot x_i y_j.$$
Tensors of order two

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► Formula using matrix representation:

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M(x, y) = x^\top My = \sum_{i,j} M_{i,j} \cdot x_i y_j .
\]

Tensors are multi-linear generalization.
Tensors of order $p$

$p$-linear functions: $T: \mathbb{R}^d \times \mathbb{R}^d \times \cdots \times \mathbb{R}^d \rightarrow \mathbb{R}$. 
Tensors of order \( p \)

\( p \)-linear functions: \( T: \mathbb{R}^d \times \mathbb{R}^d \times \cdots \times \mathbb{R}^d \to \mathbb{R} \).

- Can describe \( T \) by \( d^p \) values \( T(e_{i_1}, e_{i_2}, \ldots, e_{i_p}) =: T_{i_1, i_2, \ldots, i_p} \).
Tensors of order $p$

$p$-linear functions: $T: \mathbb{R}^d \times \mathbb{R}^d \times \cdots \times \mathbb{R}^d \to \mathbb{R}$.

- Can describe $T$ by $d^p$ values $T(e_{i_1}, e_{i_2}, \ldots, e_{i_p}) =: T_{i_1, i_2, \ldots, i_p}$.
- Identify $T$ with multi-index array $T \in \mathbb{R}^{d \times d \times \cdots \times d}$. 

$\blacksquare$
Tensors of order $p$

$p$-linear functions: $\mathbf{T} : \mathbb{R}^d \times \mathbb{R}^d \times \cdots \times \mathbb{R}^d \rightarrow \mathbb{R}$.

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- Identify $\mathbf{T}$ with multi-index array $\mathbf{T} \in \mathbb{R}^{d \times d \times \cdots \times d}$.

Formula for function value:

$$\mathbf{T}(x^{(1)}, x^{(2)}, \ldots, x^{(p)}) = \sum_{i_1,i_2,\ldots,i_p} T_{i_1,i_2,\ldots,i_p} \cdot x_{i_1}^{(1)} x_{i_2}^{(2)} \cdots x_{i_p}^{(p)}.$$
Tensors of order $p$

$p$-linear functions: $T: \mathbb{R}^d \times \mathbb{R}^d \times \cdots \times \mathbb{R}^d \to \mathbb{R}$.

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- Rank-1 tensor: $T = v^{(1)} \otimes v^{(2)} \otimes \cdots \otimes v^{(p)}$,

$$T(x^{(1)}, x^{(2)}, \ldots, x^{(p)}) = \langle v^{(1)}, x^{(1)} \rangle \langle v^{(2)}, x^{(2)} \rangle \cdots \langle v^{(p)}, x^{(p)} \rangle.$$
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$$T(x^{(1)}, x^{(2)}, \ldots, x^{(p)}) = \sum_{i_1, i_2, \ldots, i_p} T_{i_1, i_2, \ldots, i_p} \cdot x^{(1)}_{i_1} x^{(2)}_{i_2} \cdots x^{(p)}_{i_p}.$$ 

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Symmetric rank-1 tensor: $T = v^{\otimes p} = v \otimes v \otimes \cdots \otimes v$,

$$T(x^{(1)}, x^{(2)}, \ldots, x^{(p)}) = \langle v, x^{(1)} \rangle \langle v, x^{(2)} \rangle \cdots \langle v, x^{(p)} \rangle.$$
Most Tensor Problems Are NP-Hard

CHRISTOPHER J. HILLAR, Mathematical Sciences Research Institute
LEK-HENG LIM, University of Chicago

We prove that multilinear (tensor) analogues of many efficiently computable problems in numerical linear algebra are NP-hard. Our list includes: determining the feasibility of a system of bilinear equations, deciding whether a 3-tensor possesses a given eigenvalue, singular value, or spectral norm; approximating an eigenvalue, eigenvector, singular vector, or the spectral norm; and determining the rank or best rank-1 approximation of a 3-tensor. Furthermore, we show that restricting these problems to symmetric tensors does not alleviate their NP-hardness. We also explain how deciding nonnegative definiteness of a symmetric 4-tensor is NP-hard and how computing the combinatorial hyperdeterminant is NP-, #P-, and VNP-hard.
Example: rank

- Rank of $T$: smallest $r$ s.t. $T$ is sum of $r$ rank-1 tensors.
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- “Border rank” of $T$: smallest $r$ s.t. there exists sequence $(T_k)_{k \in \mathbb{N}}$ of rank-$r$ tensors with $\lim_{k \to \infty} T_k = T$. 

Define $T := x \otimes x \otimes y + x \otimes y \otimes x + y \otimes x \otimes x$, which has rank 3.

Define $T_1/\epsilon := \frac{1}{\epsilon} (x + \epsilon y) \otimes (x + \epsilon y) \otimes (x + \epsilon y) - \frac{1}{\epsilon} x \otimes x \otimes x$, which have rank 2.

For $\epsilon = 1/k$, have $\lim_{k \to \infty} T_k = T$. 

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For $\epsilon = 1/k$, have $\lim_{k \to \infty} T_k = T$. 
Aside: eigenvalue decomposition

**Recall:** every symmetric matrix $M \in \mathbb{R}^{d \times d}$ of rank $K$ has an *eigen-decomposition* (which can be efficiently computed):

$$M = \sum_{t=1}^{K} \lambda_t v_t v_t^T,$$

where $\{\lambda_t\}_{t=1}^{K}$ are the eigenvalues, and $\{v_t\}_{t=1}^{K}$ are the corresponding *eigenvectors*, which are orthonormal (i.e., orthogonal & unit length). Decomposition is unique iff $\{\lambda_t\}_{t=1}^{K}$ are distinct. (Up to sign of $v_t$s.) For (symmetric) tensors of order $p \geq 3$: an analogous decomposition is not guaranteed to exist.
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- Decomposition is unique iff $\{\lambda_t\}_{t=1}^{K}$ are distinct.
  (Up to sign of $v_t$s.)

For (symmetric) tensors of order $p \geq 3$: an analogous decomposition is **not** guaranteed to exist.
Reduction to orthonormal case

Suppose we have (estimates of) moments of the form

\[ M = \sum_{t=1}^{K} v_t \otimes v_t, \]  

(e.g., word pairs)

and

\[ T = \sum_{t=1}^{K} \lambda_t \cdot v_t \otimes v_t \otimes v_t. \]  

(e.g., word triples)

Here, we assume \( \{v_t\}_{t=1}^{K} \) are linearly independent, and \( \{\lambda_t\}_{t=1}^{K} \) are positive.
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\[ \Rightarrow \quad M \text{ is positive semidefinite of rank } K. \]
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Here, we assume \( \{v_t\}_{t=1}^{K} \) are linearly independent, and \( \{\lambda_t\}_{t=1}^{K} \) are positive.

- \( M \) is positive semidefinite of rank \( K \).
- \( M \) determines inner product system on \( \text{span} \{v_t\}_{t=1}^{K} \) s.t. \( \{v_t\}_{t=1}^{K} \) are orthonormal:

\[
\langle x, y \rangle_M := x^\top M^\dagger y.
\]
Reduction to orthonormal case

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\[ M = \sum_{t=1}^{K} \mathbf{v}_t \otimes \mathbf{v}_t, \quad \text{(e.g., word pairs)} \]

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\[ \Rightarrow M \text{ is positive semidefinite of rank } K. \]
\[ \Rightarrow M \text{ determines inner product system on } \text{span} \{\mathbf{v}_t\}_{t=1}^{K} \text{ s.t.} \]
\[ \{\mathbf{v}_t\}_{t=1}^{K} \text{ are orthonormal:} \]

\[ \langle x, y \rangle_M := x^\top M^\dagger y. \]

\[ \Rightarrow \therefore \text{Can assume } d = K \text{ and } \{\mathbf{v}_t\}_{t=1}^{d} \text{ are orthonormal.} \]

(Similar to PCA; called “whitening” in signal processing context.)
Orthogonally decomposable tensors \((d = K)\)

**Goal:** Given tensor \(T = \sum_{t=1}^d \lambda_t \cdot v_t \otimes v_t \otimes v_t \in \mathbb{R}^{d \times d \times d}\) where:

- \(\{v_t\}_{t=1}^d\) are orthonormal;
- all \(\lambda_t > 0\);

approximately recover \(\{(v_t, \lambda_t)\}_{t=1}^d\).
Exact orthogonally decomposable tensor
(Zhang & Golub, 2001)

Matching moments:

\[
\left\{ (\hat{\nu}_t, \hat{\lambda}_t) \right\}_{t=1}^{d} := \arg \min_{\left\{ (x_t, \sigma_t) \right\}_{t=1}^{d}} \left\| \mathbf{T} - \sum_{t=1}^{d} \sigma_t \cdot \mathbf{x}_t \otimes \mathbf{x}_t \otimes \mathbf{x}_t \right\|_F^2.
\]

(Here, \( \| \cdot \|_F \) is “Frobenius norm”, just like for matrices.)
Exact orthogonally decomposable tensor
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Matching moments:

\[ \{(\hat{v}_t, \hat{\lambda}_t)\}_{t=1}^{d} := \arg \min_{\{(x_t, \sigma_t)\}_{t=1}^{d}} \left\| T - \sum_{t=1}^{d} \sigma_t \cdot x_t \otimes x_t \otimes x_t \right\|_F^2. \]

(Here, \( \| \cdot \|_F \) is “Frobenius norm”, just like for matrices.)

▶ Greedy approach:

▶ Find best rank-1 approximation:

\[ (\hat{v}, \hat{\lambda}) := \arg \min_{\|x\|=1, \sigma \geq 0} \| T - \sigma \cdot x \otimes x \otimes x \|_F^2. \]

▶ “Deflate” \( T := T - \hat{\lambda} \cdot \hat{v} \otimes \hat{v} \otimes \hat{v} \) and repeat.
Exact orthogonally decomposable tensor
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Matching moments:

\[
\{(\hat{v}_t, \hat{\lambda}_t)\}_{t=1}^{d} := \underset{\{(x_t,\sigma_t)\}_{t=1}^{d}}{\operatorname{arg\,min}} \left\| T - \sum_{t=1}^{d} \sigma_t \cdot x_t \otimes x_t \otimes x_t \right\|_F^2.
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(Here, \(\| \cdot \|_F\) is “Frobenius norm”, just like for matrices.)

▶ Greedy approach:

▶ Find best rank-1 approximation:

\[
\hat{v} := \operatorname{arg\,max}_{\|x\|=1} T(x, x, x), \quad \hat{\lambda} := T(\hat{v}, \hat{v}, \hat{v}).
\]

▶ “Deflate” \(T := T - \hat{\lambda} \cdot \hat{v} \otimes \hat{v} \otimes \hat{v}\) and repeat.
Claim: Local maximizers of the function

\[ x \mapsto T(x, x, x) = \sum_{i,j,k} T_{i,j,k} \cdot x_i x_j x_k \]

(over the unit ball) are \{v_t\}_{t=1}^d, and

\[ T(v_t, v_t, v_t) = \lambda_t, \quad t \in [d]. \]
Claim: Local maximizers of the function

\[ \mathbf{x} \mapsto \mathbf{T}(\mathbf{x}, \mathbf{x}, \mathbf{x}) = \sum_{i,j,k} T_{i,j,k} \cdot x_i x_j x_k = \sum_{t=1}^{d} \lambda_t \cdot \langle \mathbf{v}_t, \mathbf{x} \rangle^3 \]

(over the unit ball) are \( \{ \mathbf{v}_t \}_{t=1}^{d} \), and

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(over the unit ball) are \( \{v_t\}_{t=1}^{d} \), and

\[ T(v_t, v_t, v_t) = \lambda_t , \quad t \in [d] . \]

Corollary: decomposition of \( T \) as \( \sum_{t=1}^{K} \lambda_t \cdot v_t^\otimes 3 \) is unique!
Proof

By linearity and orthogonality:

\[ T(v_t, v_t, v_t) = \sum_{s=1}^{d} (\lambda_s \cdot v_s^\otimes 3)(v_t, v_t, v_t) \]
Proof

By linearity and orthogonality:

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\lambda_s & \text{if } s = t \\
0 & \text{if } s \neq t
\end{cases} \]
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T(v_t, v_t, v_t) = \sum_{s=1}^{d} (\lambda_s \cdot v_s^\otimes 3)(v_t, v_t, v_t) = \sum_{s=1}^{d} \left\{ \begin{array}{ll} 
\lambda_s & \text{if } s = t \\
0 & \text{if } s \neq t 
\end{array} \right\} = \lambda_t.
\]

WLOG assume \(v_t = e_t\), so optimization problem is

\[
\max_{x \in \mathbb{R}^d} \sum_{t=1}^{d} \lambda_t x_t^3 \quad \text{s.t.} \quad \sum_{t=1}^{d} x_t^2 \leq 1.
\]
Proof

By linearity and orthogonality:

\[ T(v_t, v_t, v_t) = \sum_{s=1}^{d} (\lambda_s \cdot v_s^3)(v_t, v_t, v_t) = \sum_{s=1}^{d} \left\{ \begin{array}{ll} \lambda_s & \text{if } s = t \\ 0 & \text{if } s \neq t \end{array} \right. = \lambda_t. \]

WLOG assume \( v_t = e_t \), so optimization problem is

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\max_{x \in \mathbb{R}^d} \sum_{t=1}^{d} \lambda_t x_t^3 \quad \text{s.t.} \quad \sum_{t=1}^{d} x_t^2 \leq 1.
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If both \( x_1 \) and \( x_2 \) are non-zero, then

\[ \lambda_1 x_1^3 + \lambda_2 x_2^3 < \lambda_1 x_1^2 + \lambda_2 x_2^2 \]
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T(v_t, v_t, v_t) = \sum_{s=1}^{d} (\lambda_s \cdot v_s \otimes 3)(v_t, v_t, v_t) = \sum_{s=1}^{d} \begin{cases} 
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0 & \text{if } s \neq t
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Uniqueness of orthogonal decompositions

What we have seen so far:

1. When components \( \{v_t\}_{t=1}^d \) are linearly independent:
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Algorithm: use gradient ascent to find all of the local maximizers, which are exactly $v_t$. (Can use "deflation" to remove components from $T$ that you've already found.)
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Application to topic model parameters

Probabilities of word triples as third-order tensor:

\[ T = \sum_{t=1}^{K} w_t P_t \otimes P_t \otimes P_t = \sum_{t=1}^{K} v_t \otimes v_t \otimes v_t \]

for \( v_t = w_t^{1/3} P_t \).
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Recap

- Parameters of topic model for single-topic documents (satisfying linear independence condition) can be efficiently recovered from distribution of three-word documents.
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- Two-word documents not sufficient (without further assumptions).
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- Two-word documents not sufficient (without further assumptions).

- Variational characterization of orthogonally decomposable tensors leads to simple and efficient algorithms!
Illustrative empirical results

- Corpus: 300,000 New York Times articles.
- Vocabulary size: 102,660 words.
- Set number of topics $K := 50$.

**Model predictive performance:**
$\approx 4-8 \times$ speed-up over Gibbs sampling for LDA; comparable to “FastLDA” (Porteous, Newman, Ihler, Asuncion, Smyth, & Welling, 2008).

![Graph showing log loss vs training time for different methods]

- **Method-of-moments**
- **Gibbs sampling**
- **FastLDA**

Log loss

Training time ($\times 10^4$ sec)
**Illustrative empirical results**

**Sample topics:** (showing top 10 words for each topic)

<table>
<thead>
<tr>
<th>Econ.</th>
<th>Baseball</th>
<th>Edu.</th>
<th>Health care</th>
<th>Golf</th>
</tr>
</thead>
<tbody>
<tr>
<td>sales</td>
<td>run</td>
<td>school</td>
<td>drug</td>
<td>player</td>
</tr>
<tr>
<td>economic</td>
<td>inning</td>
<td>student</td>
<td>patient</td>
<td>tiger_wood</td>
</tr>
<tr>
<td>consumer</td>
<td>hit</td>
<td>teacher</td>
<td>million</td>
<td>won</td>
</tr>
<tr>
<td>major</td>
<td>game</td>
<td>program</td>
<td>company</td>
<td>shot</td>
</tr>
<tr>
<td>home</td>
<td>season</td>
<td>official</td>
<td>doctor</td>
<td>play</td>
</tr>
<tr>
<td>indicator</td>
<td>home</td>
<td>public</td>
<td>companies</td>
<td>round</td>
</tr>
<tr>
<td>weekly</td>
<td>right</td>
<td>children</td>
<td>percent</td>
<td>win</td>
</tr>
<tr>
<td>order</td>
<td>games</td>
<td>high</td>
<td>cost</td>
<td>tournament</td>
</tr>
<tr>
<td>claim</td>
<td>dodger</td>
<td>education</td>
<td>program</td>
<td>tour</td>
</tr>
<tr>
<td>scheduled</td>
<td>left</td>
<td>district</td>
<td>health</td>
<td>right</td>
</tr>
</tbody>
</table>
Illustrative empirical results

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<table>
<thead>
<tr>
<th>Invest.</th>
<th>Election</th>
<th>auto race</th>
<th>Child’s Lit.</th>
<th>Afghan War</th>
</tr>
</thead>
<tbody>
<tr>
<td>percent</td>
<td>al_gore</td>
<td>car</td>
<td>book</td>
<td>taliban</td>
</tr>
<tr>
<td>stock</td>
<td>campaign</td>
<td>race</td>
<td>children</td>
<td>attack</td>
</tr>
<tr>
<td>market</td>
<td>president</td>
<td>driver</td>
<td>ages</td>
<td>afghanistan</td>
</tr>
<tr>
<td>fund</td>
<td>george_bush</td>
<td>team</td>
<td>author</td>
<td>official</td>
</tr>
<tr>
<td>investor</td>
<td>bush</td>
<td>won</td>
<td>read</td>
<td>military</td>
</tr>
<tr>
<td>companies</td>
<td>clinton</td>
<td>win</td>
<td>newspaper</td>
<td>u_s</td>
</tr>
<tr>
<td>analyst</td>
<td>vice</td>
<td>racing</td>
<td>web</td>
<td>united_states</td>
</tr>
<tr>
<td>money</td>
<td>presidential</td>
<td>track</td>
<td>writer</td>
<td>terrorist</td>
</tr>
<tr>
<td>investment</td>
<td>million</td>
<td>season</td>
<td>written</td>
<td>war</td>
</tr>
<tr>
<td>economy</td>
<td>democratic</td>
<td>lap</td>
<td>sales</td>
<td>bin</td>
</tr>
</tbody>
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<table>
<thead>
<tr>
<th>Web</th>
<th>Antitrust</th>
<th>TV</th>
<th>Movies</th>
<th>Music</th>
</tr>
</thead>
<tbody>
<tr>
<td>com</td>
<td>court</td>
<td>show</td>
<td>film</td>
<td>music</td>
</tr>
<tr>
<td>www</td>
<td>case</td>
<td>network</td>
<td>movie</td>
<td>song</td>
</tr>
<tr>
<td>site</td>
<td>law</td>
<td>season</td>
<td>director</td>
<td>group</td>
</tr>
<tr>
<td>web</td>
<td>lawyer</td>
<td>nbc</td>
<td>play</td>
<td>part</td>
</tr>
<tr>
<td>sites</td>
<td>federal</td>
<td>cb</td>
<td>character</td>
<td>new_york</td>
</tr>
<tr>
<td>information</td>
<td>government</td>
<td>program</td>
<td>actor</td>
<td>company</td>
</tr>
<tr>
<td>online</td>
<td>decision</td>
<td>television</td>
<td>show</td>
<td>million</td>
</tr>
<tr>
<td>mail</td>
<td>trial</td>
<td>series</td>
<td>movies</td>
<td>band</td>
</tr>
<tr>
<td>internet</td>
<td>microsoft</td>
<td>night</td>
<td>million</td>
<td>show</td>
</tr>
<tr>
<td>telegram</td>
<td>right</td>
<td>new_york</td>
<td>part</td>
<td>album</td>
</tr>
</tbody>
</table>

*etc.*
Computation

**Caveat**: forming and computing with a third-order tensor $T$ generally requires cubic space.
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Learning algorithms

- Estimation via **method-of-moments**:
  1. *Estimate* distribution of three-word documents $\rightarrow \hat{T}$
    
    *(empirical moment tensor)*.
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**Issues**:

1. **Accuracy of moment estimates**?
   - Can more reliably estimate lower-order moments; distribution-specific sample complexity bounds.
2. **Robustness of (approximate) tensor decomposition**?
   - In some sense, more stable than matrix eigen-decomposition (Mu, H., & Goldfarb, 2015).
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   - Next: Moment decompositions for other models.
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Next: Moment decompositions for other models.
2. Moment decompositions for other models
Some examples of usable moment decompositions.

1. Two classical mixture models.
Mixture model #1: Mixtures of spherical Gaussians

\[ H \sim \text{Categorical}(\pi_1, \pi_2, \ldots, \pi_K) \text{ (hidden)}; \]
\[ X \mid H = t \sim \text{Normal}(\mu_t, \sigma_t^2 I_d), \quad t \in [K]. \]
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Generative process:

\[ X = Y + \sigma Z \]

where \( \Pr(Y = \mu_t) = \pi_t \), and
\[ Z \sim \text{Normal}(0, I_d) \quad \text{(indep. of } Y). \]
Using moments for spherical Gaussian mixtures

We’ll see two ways to use low-order moments.
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First- and second-order moments:

\[ \mathbb{E}(X) \in \mathbb{R}^d \quad \text{and} \quad \mathbb{E}(X \otimes X) \in \mathbb{R}^{d \times d}. \]
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**Claim (Vempala & Wang, 2002):**
Span of top \( K \) eigenvectors of \( \mathbb{E}(X \otimes X) \) contains \( \{\mu_t\}_{t=1}^K \).
\((K\text{-dimensional Principal Component Analysis (PCA) subspace.})\)
Proof

**Key fact:** $k$-dimensional PCA subspace (based on $\mathbb{E}(X \otimes X)$) captures as much of overall variance as any other $k$-dim. subspace.
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\mathbb{E}(X \otimes X) = \mu_1 \otimes \mu_1 + \sigma^2 I_d.
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Variance in direction \( \mathbf{v} \) (with \( \|\mathbf{v}\| = 1 \)):

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\mathbf{v}^\top \mathbb{E}(X \otimes X) \mathbf{v}
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Variance in direction $\nu$ (with $\|\nu\| = 1$):

\[
\nu^\top \mathbb{E}(X \otimes X) \nu = \nu^\top (\mu_1 \otimes \mu_1 + \sigma^2 I_d) \nu = (\nu^\top \mu_1)^2 + \sigma^2.
\]

**Best direction** (1-dim. PCA subspace): $\nu = \pm \mu_1 / \|\mu_1\|$. 
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$K = 1$ (just a single Gaussian):
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![Diagram of PCA subspace.]
Proof (continued)

**Key fact:** $k$-dimensional PCA subspace (based on $\mathbb{E}(X \otimes X)$) captures as much of overall variance as any other $k$-dim. subspace.

$\blacktriangleright$ $K = 1$ (just a single Gaussian):

What is the $k$-dimensional PCA subspace?

![Diagram of $\mathbb{R}^d$ with origin and $\mu_1$]

**Answer:** any $k$-dim. subspace containing $\mu_1$. 
Proof (continued)

**Key fact:** $k$-dimensional PCA subspace (based on $\mathbb{E}(X \otimes X)$) captures as much of overall variance as any other $k$-dim. subspace.

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How does this help with learning mixtures of Gaussians?
Use of moments for mixtures of spherical Gaussians

**Separation** *(Dasgupta, 1999):*

# standard deviations between component means

\[
    \text{sep} := \min_{i \neq j} \frac{\| \mu_i - \mu_j \|}{\sigma}.
\]

*(Dasgupta & Schulman, 2000):* Distance-based clustering (e.g., EM) works when \( \text{sep} \gtrsim d^{1/4} \).

*(Vempala & Wang, 2002):* Problem becomes \( K \)-dimensional via PCA (assume \( K \leq d \)). Required separation reduced to \( \text{sep} \gtrsim K^{1/4} \).

Third-order moments identify the mixture distribution when \( \{\mu_t\}_{t=1}^K \) are lin. indpt.; sep may be arbitrarily close to zero. *(Belkin & Sinha, 2010; Moitra & Valiant, 2010):* General Gaussians & no minimum sep, but \( K \)th-order moments.
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Third-order moments of spherical Gaussian mixtures

**Generative process:**

\[ X = Y + \sigma Z \]

where \( \Pr(Y = \mu_t) = \pi_t \), and \( Z \sim \text{Normal}(0, I_d) \), \( Y \perp Z \).

Third-order moment tensor:

\[ \mathbb{E} \left( X^{\otimes 3} \right) = \mathbb{E} \left( \{Y + \sigma Z\}^{\otimes 3} \right) \]
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\]

(Above, \( \mu = \mathbb{E}(X) \) and \( \tau(\mu) \) is a third-order tensor involving only \( \mu \).)

**Exercise:** find explicit formula for \( \tau(\mu) \).
Third-order moments of spherical Gaussian mixtures

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\left( Y \otimes Z \otimes Z + Z \otimes Y \otimes Z + Z \otimes Z \otimes Y \right)
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Third-order moments of spherical Gaussian mixtures

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Tensor decomposition for spherical Gaussian mixtures
(H. & Kakade, 2013)

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**Claim:** \( \mu \) & \( \sigma^2 \) are simple functions of \( \mathbb{E}(X) \) & \( \mathbb{E}(X \otimes X) \).
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Claim: If $\{\mu_t\}_{t=1}^K$ are linearly independent and all $\pi_t > 0$, then $\{(\mu_t, \pi_t)\}_{t=1}^K$ are identifiable from

$$T := \mathbb{E}(X \otimes^3) - \sigma^2 \tau(\mu) = \sum_{t=1}^{K} \pi_t \cdot \mu_t^{\otimes 3}.$$ 

Can use tensor decomposition to recover $\{(\mu_t, \pi_t)\}_{t=1}^K$ from $T$. 
Even more Gaussian mixtures

**Note**: Linear independence condition on $\{\mu_t\}_{t=1}^K$ requires $K \leq d$. 
Even more Gaussian mixtures

Note: Linear independence condition on \( \{\mu_t\}_{t=1}^K \) requires \( K \leq d \).

▶ (Anderson, Belkin, Goyal, Rademacher, & Voss, 2014),
(Bhaskara, Charikar, Moitra, & Vijayaraghavan, 2014)
Mixtures of \( d^{O(1)} \) Gaussians (w/ simple or known covariance)
via smoothed analysis and \( O(1) \)-order moments.
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  Mixtures of $d^{O(1)}$ Gaussians (w/ simple or known covariance) via smoothed analysis and $O(1)$-order moments.

- (Ge, Huang, & Kakade, 2015)
  Also with unknown covariances of arbitrary shape.
Mixture model #2: Mixtures of linear regressions

\[ H \sim \text{Categorical}(\pi_1, \pi_2, \ldots, \pi_K) \] (hidden);
\[ X \sim \text{Normal}(\mu, \Sigma); \]
\[ Y \mid H = t, X = x \sim \text{Normal}(\langle \beta_t, x \rangle, \sigma^2). \]
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Use of moments for mixtures of linear regressions

**Second-order moments** (assume $X \sim \text{Normal}(0, I_d)$):

$$
\mathbb{E}(Y^2 X X^\top) = 2 \sum_{t=1}^{K} \pi_t \cdot \beta_t \beta_t^\top + \left( \sigma^2 + \sum_{t=1}^{K} \pi_t \cdot \| \beta_t \|^2 \right) I_d.
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▶ Span of top $K$ eigenvectors of $\mathbb{E}(Y^2XX^\top)$ contains $\{\beta_t\}_{t=1}^{K}$.
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- Using **Stein's identity (1973)**, similar approach works for GLMs (Sun, Ioannidis, & Montanari, 2013).
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**Tensor decomposition approach:**
Can recover parameters $\{(\beta_t, \pi_t)\}_{t=1}^{K}$ with higher-order moments (Chaganty & Liang, 2013; Yi, Caramanis, & Sanghavi, 2014, 2016).
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Also for GLMs, via Stein’s identity (Sedghi & Anandkumar, 2014).
Recap: mixtures of Gaussians and linear regressions

- Parameters of Gaussian mixture models and related models (satisfying linear independence condition) can be efficiently recovered from $O(1)$-order moments.
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Recap: mixtures of Gaussians and linear regressions

- **Parameters of Gaussian mixture models and related models** (satisfying linear independence condition) can be efficiently recovered from $O(1)$-order moments.

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**Next:** Multi-view approach to finding usable moments.
Multi-view interpretation of topic model

Recall: Topic model for single-topic documents

$K$ topics (dists. over words) $\{P_t\}_{t=1}^K$. Pick topic $H = t$ with prob. $w_t$ (hidden).

Word tokens $X_1, X_2, \ldots, X_L \sim P_H$. 

Key property: $X_1, X_2, \ldots, X_L$ conditionally independent given $H$. Each word token $X_i$ provides new "view" of hidden variable $H$. Some previous analyses:

▶ (Chaudhuri, Kakade, Livescu, & Sridharan, 2009) Multi-view Gaussian mixture models.
Multi-view interpretation of topic model

**Recall:** Topic model for single-topic documents

- **Diagram:**
  - Root node: $H$
  - Children nodes: $X_1, X_2, \ldots, X_L$
  - Edges: $H \rightarrow X_1, H \rightarrow X_2, \ldots, H \rightarrow X_L$

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**Some previous analyses:**

- (Blum & Mitchell, 1998)
  *Co-training* in semi-supervised learning.

- (Chaudhuri, Kakade, Livescu, & Sridharan, 2009)
  Multi-view Gaussian mixture models.
Multi-view mixture model

View 1: $X_1$  View 2: $X_2$  View 3: $X_3$
Multi-view mixture model

$H$

$X_1$ $X_2$ $X_3$

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Tensor decomposition approach works in this asymmetric case as long as $\{\mu_j(t)\}_{t=1}^K$ are lin. indpt. for each $j$, and all $\pi_t > 0$. 
Multi-view mixture model

\[ \mathbb{E}(X_1 \otimes X_2 \otimes X_3) = \sum_{t=1}^{K} \pi_t \cdot \mu^{(1)}_t \otimes \mu^{(2)}_t \otimes \mu^{(3)}_t \]

where \( \mu^{(i)}_t = \mathbb{E}[X_i \mid H = t] \),
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Examples of multi-view mixture models
(Mossel & Roch, 2006; Anandkumar, H., & Kakade, 2012)

1. Mixtures of high-dimensional product distributions.
   (E.g., mixtures of axis-aligned Gaussians, other topic models.)
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2. Hidden Markov models.

\[
\begin{array}{c}
H_1 \rightarrow H_2 \rightarrow H_3 \\
X_1 \rightarrow X_2 \rightarrow X_3
\end{array}
\rightarrow
\begin{array}{c}
H_2 \\
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   - $X_1, X_2, X_3$: genes of three extant species.
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Next: Single index models.
Single-index models

\[
\begin{align*}
X & \sim \text{Normal}(0, I); \\
Y | X = x & \sim \text{Normal}(g(\langle \beta, x \rangle), \sigma^2).
\end{align*}
\]

Here, \( g : \mathbb{R} \rightarrow \mathbb{R} \) is the \textit{link function}.
Single-index models

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- **Phase retrieval** (real signals): assume \( g(z) = z^2 \).
- **1-bit compressed sensing**: assume \( g(z) = \text{sign}(z) \).
- **Isotonic regression**: assume \( g \) is monotone (e.g., \( g' \geq 0 \)).
- **Convex regression**: assume \( g \) is convex (e.g., \( g'' \geq 0 \)).
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When \( g \) is unknown, model is generally called **single-index model**.
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When \( g \) is unknown, model is generally called \textit{single-index model}.

**Semi-parametric estimation**: regard \( g \) as nuisance parameter; focus on estimating \( \beta \).
Aside: symmetric tensors and homogeneous polynomials

Recall formula for tensor function value:

\[ T(x^{(1)}, \ldots, x^{(p)}) = \sum_{i_1, \ldots, i_p} T_{i_1, \ldots, i_p} \cdot x^{(1)}_{i_1} \cdots x^{(p)}_{i_p}. \]
Aside: symmetric tensors and homogeneous polynomials

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\[ T(x^{(1)}, \ldots, x^{(p)}) = \sum_{i_1, \ldots, i_p} T_{i_1, \ldots, i_p} \cdot x^{(1)}_{i_1} \cdots x^{(p)}_{i_p}. \]

If \( T \) is symmetric (i.e., \( T_{i_1, \ldots, i_p} = T_{\pi(i_1), \ldots, \pi(i_p)} \) for any permutation \( \pi \)), then evaluating at \( x^{(1)} = \cdots = x^{(p)} = x \) gives

\[ T(x, \ldots, x) = c_p \sum_{i_1 \leq \cdots \leq i_p} T_{i_1, \ldots, i_p} \cdot x_{i_1} \cdots x_{i_p}, \]

which is just the formula for a degree-\( p \) homogeneous polynomial.
Aside: symmetric tensors and homogeneous polynomials

Recall formula for tensor function value:

$$T(x^{(1)}, \ldots, x^{(p)}) = \sum_{i_1, \ldots, i_p} T_{i_1, \ldots, i_p} \cdot x_{i_1}^{(1)} \cdots x_{i_p}^{(p)}.$$ 

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$$T(x, \ldots, x) = c_p \sum_{i_1 \leq \cdots \leq i_p} T_{i_1, \ldots, i_p} \cdot x_{i_1} \cdots x_{i_p},$$

which is just the formula for a degree-$p$ homogeneous polynomial.

$p$-th order symmetric tensors $\simeq$ degree-$p$ homogeneous polynomials.
Let $H_p : \mathbb{R} \to \mathbb{R}$ denote the degree-$p$ Hermite polynomial.

Assume (for $Z \sim \text{Normal}(0, 1)$):

$\triangleright$ $\mathbb{E}[g(Z)^2] = 1$ (normalization—this is WLOG);

$\triangleright$ $\mathbb{E}[g'(Z)^2] \geq \epsilon$ (necessary for identifiability);

$\triangleright$ $g$ is smooth and $\mathbb{E}[g''(Z)^2] = O(1)$.
Using orthogonal polynomials
(Dudeja & H., 2018)

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There exists $p = O(1/\epsilon)$ such that

$$
\mathbb{E}[Y H_p(\langle \mathbf{v}, \mathbf{X} \rangle)] = (\lambda \beta \otimes^p)(\mathbf{v}), \quad \mathbf{v} \in \mathbb{R}^d
$$

for some $\lambda \neq 0$ with $|\lambda| = \Omega(\epsilon/\sqrt{p})$. 
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$\Rightarrow$ Get efficient algorithms for semi-parametric estimation of single-index model parameters, for very general link functions.
Recap

- Parameters of many latent variable models (satisfying non-degeneracy conditions) can be efficiently recovered from $O(1)$-order moments.
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- Exploit distributional properties, multi-view structure, and other structure to determine usable moments.

- Estimation via **method-of-moments**:
  1. *Estimate* moments $\rightarrow$ empirical moment tensor $\hat{T}$.
  2. *Approximately decompose* $\hat{T} \rightarrow$ parameter estimate $\hat{\theta}$. 
3. Error analysis
Moment estimates

Estimation of $\mathbb{E}[X^{\otimes 3}]$ (say) from iid sample $\{x_i\}_{i=1}^n$:

$$\hat{\mathbb{E}}[X^{\otimes 3}] := \frac{1}{n} \sum_{i=1}^n x_i^{\otimes 3}.$$
Moment estimates

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$$\hat{\mathbb{E}}[X^{\otimes 3}] := \frac{1}{n} \sum_{i=1}^n x_i^{\otimes 3}.$$ 

Inevitably expect error of order $n^{-1/2}$ in some norm, e.g.,

$$\|T\| := \sup_{\|x\|=\|y\|=\|z\|=1} T(x, y, z) \quad \text{(injective/“spectral” norm)},$$

$$\|T\|_F := \left( \sum_{i,j,k} T_{i,j,k}^2 \right)^{1/2} \quad \text{(Frobenius norm)}.$$
Nearly orthogonally decomposable tensor
(Mu, H., & Goldfarb, 2015)

Let $\varepsilon = \|E\|$ for $E := \hat{T} - T$.

Claim: Let $\hat{v} := \arg\max_{\|x\|=1} \hat{T}(x, x, x)$ and $\hat{\lambda} := \hat{T}(\hat{v}, \hat{v}, \hat{v})$.
Then

$$|\hat{\lambda} - \lambda_t| \leq \varepsilon, \quad \|\hat{v} - v_t\| \leq O\left(\frac{\varepsilon}{\lambda_t} + \left(\frac{\varepsilon}{\lambda_t}\right)^2\right)$$

for some $t \in [d]$ with $\lambda_t \geq \max_{t'} \lambda_{t'} - 2\varepsilon$. 
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for some \( t \in [d] \) with \( \lambda_t \geq \max_{t'} \lambda_{t'} - 2\varepsilon \).

Many efficient algorithms for solving this approximately, when \( \varepsilon \) is small enough, like \( 1/d \) or \( 1/\sqrt{d} \) (e.g., Anandkumar, Ge, H., Kakade, & Telgarsky, 2014; Ma, Shi, & Steurer, 2016).
Recall: greedy decomposition  
(Zhang & Golub, 2001)

Matching moments:

\[
\{(\hat{\mathbf{v}}_t, \hat{\lambda}_t)\}_{t=1}^d := \arg \min_{\{(x_t, \sigma_t)\}_{t=1}^d} \left\| \mathbf{T} - \sum_{t=1}^d \sigma_t \cdot \mathbf{x}_t \otimes \mathbf{x}_t \otimes \mathbf{x}_t \right\|_F^2 .
\]
Recall: greedy decomposition

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\]

▶ Greedy approach:
  ▶ Find best rank-1 approximation:

\[
(\hat{v}, \hat{\lambda}) := \arg \min_{\| x \| = 1, \sigma \geq 0} \left\| T - \sigma \cdot x \otimes x \otimes x \right\|_F^2.
\]

▶ “Deflate” \( T := T - \hat{\lambda} \cdot \hat{v} \otimes \hat{v} \otimes \hat{v} \) and repeat.
Recall: greedy decomposition

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Matching moments:

\[ \{(\hat{v}_t, \hat{\lambda}_t)\}_{t=1}^d := \arg \min_{\{(x_t, \sigma_t)\}_{t=1}^d} \left\| T - \sum_{t=1}^d \sigma_t \cdot x_t \otimes x_t \otimes x_t \right\|_F^2. \]

▶ Greedy approach:

▶ Find best rank-1 approximation:

\[ \hat{v} := \arg \max_{\|x\|=1} T(x, x, x), \quad \hat{\lambda} := T(\hat{v}, \hat{v}, \hat{v}). \]

▶ “Deflate” \( T := T - \hat{\lambda} \cdot \hat{v} \otimes \hat{v} \otimes \hat{v} \) and repeat.
Errors from deflation

(For simplicity, assume $\lambda_t = 1$ for all $t$, so $T = \sum_t v_t^\otimes 3$.)

First greedy step:
Rank-1 approx. $\hat{v}_1^\otimes 3$ to $\hat{T}$ satisfies $\|\hat{v}_1 - v_1\| \leq \varepsilon$ (say).
Errors from deflation

(For simplicity, assume $\lambda_t = 1$ for all $t$, so $T = \sum_t v_t^\otimes 3$.)

**First greedy step:**
Rank-1 approx. $\hat{v}_1^\otimes 3$ to $\hat{T}$ satisfies $\|\hat{v}_1 - v_1\| \leq \varepsilon$ (say).

**Deflation:** To find next $v_t$, use

$$\hat{T} - \hat{v}_1^\otimes 3 = T + E - \hat{v}_1^\otimes 3$$

$$= \sum_{t=2}^{d} v_t^\otimes 3 + E + (v_1^\otimes 3 - \hat{v}_1^\otimes 3).$$
(For simplicity, assume $\lambda_t = 1$ for all $t$, so $T = \sum_t v_t^{\otimes 3}$.)

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Now error seems to have doubled (i.e., of size $2\varepsilon$) ...
Effect of deflation errors

For any unit vector \( x \) orthogonal to \( v_1 \):

\[
\left\| \frac{1}{3} \nabla_x \left\{ \left( v_1 \otimes^3 - \hat{v}_1 \otimes^3 \right)(x, x, x) \right\} \right\| = \left\| \langle v_1, x \rangle^2 v_1 - \langle \hat{v}_1, x \rangle^2 \hat{v}_1 \right\|
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Effect of deflation errors

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So effect of errors (original and from deflation) $E + \left( v_1 \otimes^3 - \hat{v}_1 \otimes^3 \right)$ in directions orthogonal to $v_1$ is $(1 + o(1))\varepsilon$ rather than $2\varepsilon$. 
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$$= \langle \hat{v}_1, x \rangle^2$$

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So effect of errors (original and from deflation) $E + \left( v_1 \otimes^3 - \hat{v}_1 \otimes^3 \right)$ in directions orthogonal to $v_1$ is $(1 + o(1))\varepsilon$ rather than $2\varepsilon$.

- Deflation errors have lower-order effect on finding other $v_t$.

(Analogous statement for deflation with matrices does not hold.)
Summary

- Using method-of-moments with low-order moments, can efficiently estimate parameters for many models.
  - Exploit distributional properties, multi-view structure, and other structure to determine usable moments tensors.
  - Some efficient algorithms for carrying out the tensor decomposition to obtain parameter estimates.

Many issues to resolve!

- Handle model misspecification, increase robustness.
- General methodology.
- Incorporate general prior knowledge and interactive feedback.
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- Using method-of-moments with low-order moments, can efficiently estimate parameters for many models.
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Acknowledgements

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Further reading:


¡Gracias!