Learning Binary Relations

Presented by Alan Duan

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We call this a **binary relation**.

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2. In binary relations, the order matters.

Representing a Binary Relations

- $n \times m$ binary matrix

	Topics in Learning Theory	Machine Learning	Operating System
Alan	1	0	0
Bob	1	1	0
Cathy	0	0	1
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- 2-column table

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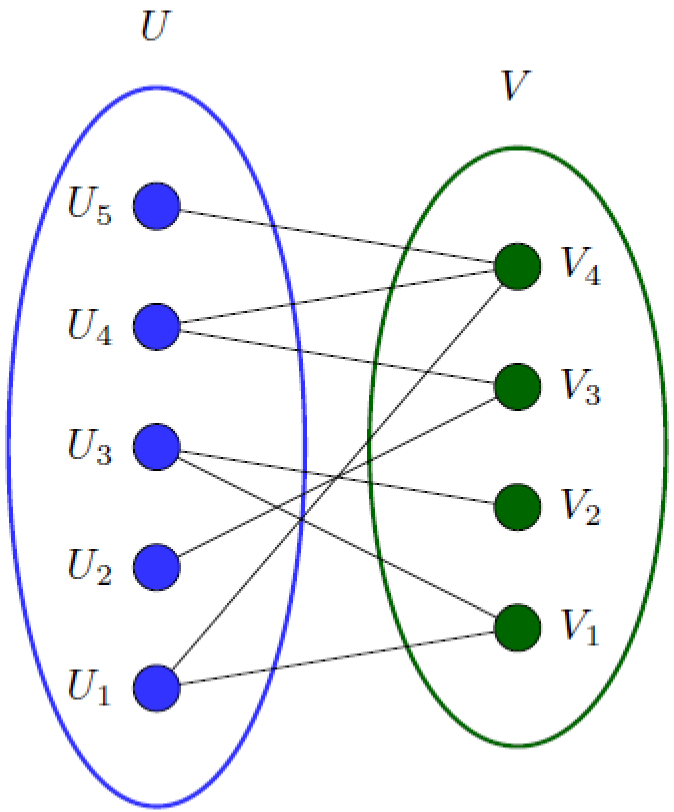
Representing a Binary Relations (cont'd)

- Bipartite graph



Learning Binary Relations

Setting

We are learning binary relations between two set A and B represented by predicate P. Denote $|A| = n$ and $|B| = m$.

In each trial t:

- learner is given an unlabeled pair of object $x_t = (a_t, b_t)$, where $a_t \in A, b_t \in B$
- learner predicts $\hat{y}_t = 0$ or 1
- reveals the answer y_t
- if answer and prediction are different, record it as a *mistake*

Goal: Minimize the number of incorrect predictions

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Note :

1. In this presentation, we will use these notation from concept learning interchangably from time to time.
2. We will see what is special about learning binary relations in a bit!

Learning Binary Relations

A few more terms

Let $\mathcal{X}$ be a finite learning domain. Let C be a concept class over $\mathcal{X}$.

A learner is *consistent* if, on every trial, there exists some concept $c \in C$ such that:

$$c(x_k) = \begin{cases} \hat{y}_t, & \text{if } k = t \\ y_k, & \text{if } k = 1, \dots, t-1 \end{cases}$$

A *query sequence* $\pi = \langle x_1, x_2, \dots, x_{|\mathcal{X}|} \rangle$ is a permutation of $\mathcal{X}$, where $x_t \in \mathcal{X}$ is the instance presented to the learner at the t^{th} trial.

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Who determines the query sequence?

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- Director Agnostic: we want some mistake bounds regardless of the director.

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- Director Agnostic: we want some mistake bounds regardless of the director.
- Self-directed: the learner itself chooses π.
- Teacher-directed: A teacher who knows the target relation and wants to minimize the learner's mistakes by choosing π; Teacher can choose x_t with the knowledge of 1) target relation, 2) $x_1, \ldots, x_{t-1}$, 3) $\hat{y}_1, \ldots, \hat{y}_{t-1}$.

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- Adversary-directed: An adversary who tries to maximize the learner's mistakes, knows the learner's algorithm and has unlimited computing power, chooses π.

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- Adversary-directed: An adversary who tries to maximize the learner's mistakes, knows the learner's algorithm and has unlimited computing power, chooses π.

For teacher-directed setting, we want to consider **worst case mistake bound** over all consistent learners. (why?)

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If there's no structure, we can't do any better than random guessing.

What can be a natural structure?

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Equiv : If we represent this binary relation using a $n \times m$ matrix, how many possible *row type* could the row for Alan be?

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Alan	?	?	$\ldots$	?

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We use k to represent the distinct row types in the matrix. We call this type of relation *k-binary-relations*.

General bounds applied to all directors

Theorem 1 (Lower Bound) For any $0 < \beta \leq 1$, any prediction algorithm makes at least $(1-\beta)km + n\lfloor \log(\beta k) \rfloor - (1-\beta)k\lfloor \log(\beta k) \rfloor$ mistakes regardless of the query sequence.

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$$2^p + q = k$$

Set $2^p = \beta k, q = (1-\beta)k$, we can get $p = \lfloor \log(\beta k) \rfloor$, $q = (1-\beta)k$.

The mistake bound: $(1-\beta)k \cdot m + \lfloor \log(\beta k) \rfloor \cdot n - (1-\beta)k \lfloor \log(\beta k) \rfloor$.

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Note : Halving algorithm (in general) can be computationally expensive!

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For the first row:

- Guess all entries. Record it as the first row type.

For the rest rows:

- Predict row i, column j's value according to a majority vote of the recorded row templates that are consistent with row i
- If no such consistent template exists, guess all the rest entries in row i, and record it as a new type. $\hat{k} = \hat{k} + 1$.

How many mistakes have we made?

- For each new row template, we make at most m on each. Total is km.
- For each of the rest rows, we make at most $\lfloor \log \hat{k} \rfloor \leq \lfloor \log k \rfloor$ mistakes. The total is $(n-k)\lfloor \log k \rfloor$.
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Note :

1. Similar flavour as the halving algorithm -- but computationally tractable.
2. Do not need to know k a priori.
3. This bound is within a constant factor of the general lower bound (Theorem 1).

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Then, for the rest of $(n - k)$ rows, the teacher presents $(k - 1)$ entries to distinguish it from the incorrect row types.

After this, for the rest of $(n - k)$ rows, its row type can be uniquely identified, and no more mistakes will be made.

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In total, the learner makes at most $km + (n-k)(k-1)$ mistakes.

Teacher-directed

Theorem 5 (Lower Bound) The number of mistakes made with a helpful teacher as the director is at least $\min\{nm, km + (n-k)(k-1)\}$.

Proof:

$$\begin{array}{r}
\text{5 row types}\left\{\begin{matrix}
0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0
\end{matrix}\right.\\
\begin{matrix}
0&0&0&0&0&0&0&0&0\\
&&&&\vdots&&&&\\
0&0&0&0&0&0&0&0&0
\end{matrix}
\end{array}$$

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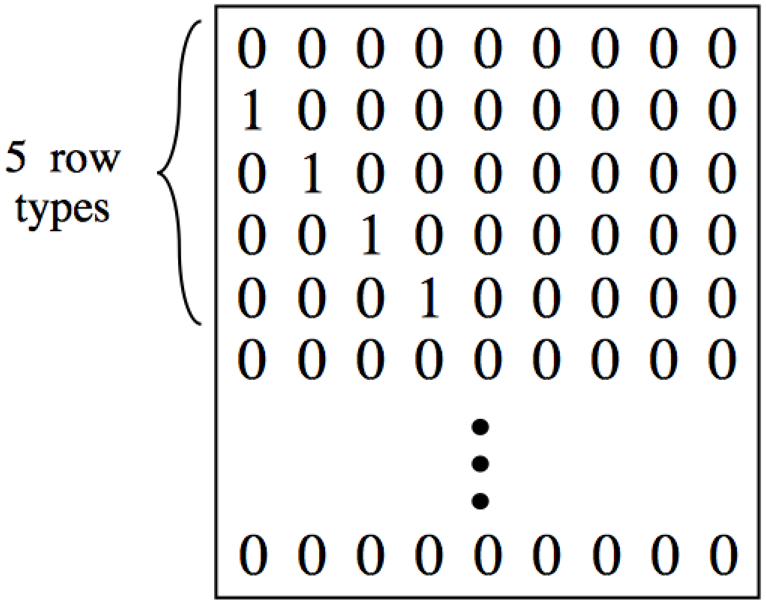
For the rest of the rows:

When $(m+1) \geq k$: we need to know all first $k-1$ columns to uniquely identify the row type.

When $(m+1) < k$: we need to know all m columns to uniquely identify the row type.

Proof:



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Adding up, the mistake bound is $\min\{km + (n-k)m, km + (n-k)(k-1)\}$.

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Question: Recall that the mistake bound for learner director is $km + (n-k)\lfloor \log k \rfloor$ **, while teacher-directed bound is** $km + (n-k)(k-1)$ **. Why is it even worse?**

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Teacher-directed case apply to all consistent learners!

Teacher-directed

Question: Recall that the mistake bound for learner director is $km + (n-k)\lfloor \log k \rfloor$, while teacher-directed bound is $km + (n-k)(k-1)$. Why is it even worse?

Teacher-directed case apply to all consistent learners!

A consistent learner may do *minority-vote* instead of *majority-vote*.

Adversary-directed

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Adding up the number of mistakes, we get the desired bound.

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Theorem 7 (Upper Bound when k=2) There exists a polynomial prediction algorithm that makes at most $2m + n - 2$ mistakes against adversary-selected query sequence when $k = 2$.

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Proof: Let's do it on board!

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This is left as an exercise.

Conclusion and Takeaways

1. In previous lectures, we usually focus on learner's algorithm, and assume the environment (director) as the worst case (adversary). It turns out to be not true in many real life cases. Maybe the director is trying to help learner to learn. And in those cases, we can indeed improve learner's performance.

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3. Some learning on proof technique: to prove an upper bound, we can prove by showing an algorithm that satisfies the bound; to prove a lower bound, we can prove by showing there exists an adversary setting that all algorithms make at least this amount of mistake.

Reference

1. Learning Binary Relations and Total Orders, *Sally A. Goldman, Ronald L. Rivest, and Robert E. Schapire*, SIAM Journal on Computing 1993 22:5, 1006-1034.