**Weierstrass approximation theorem**

COMS 4995-1 Spring 2020 (Daniel Hsu)

**Theorem** (Weierstrass approximation theorem). Suppose $f : [0, 1] \rightarrow \mathbb{R}$ is continuous. For any $\epsilon > 0$, there exists a polynomial $p$ such that

$$\sup_{x \in [0,1]} |f(x) - p(x)| \leq \epsilon.$$  

**Proof.** Since $f$ is continuous on $[0, 1]$, it is uniformly continuous. This means that for any $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that $|f(x) - f(y)| < \epsilon/2$ for all $x, y \in [0, 1]$ satisfying $|x - y| < \delta_\epsilon$. Let us fix an $\epsilon > 0$ and such a corresponding $\delta_\epsilon > 0$.

Let $r$ be any positive integer such that $r \geq \frac{\|f\|_\infty}{\delta_\epsilon^2}$. Define the Bernstein polynomials

$$b_{k,r}(x) = \Pr(S_{r,x} = k) = \binom{r}{k} x^k (1 - x)^{r-k}$$

where $S_{r,x} \sim \text{Binom}(r, x)$. Let $p(x) := \sum_{k=0}^r f\left(\frac{k}{r}\right) b_{k,r}(x)$, which is a degree-$r$ polynomial. Then, for any $x \in [0, 1],$

$$|p(x) - f(x)|$$

$$= \left| \sum_{k=0}^r \left( f\left(\frac{k}{r}\right) - f(x) \right) b_{k,r}(x) \right|$$

$$\leq \sum_{|k-rx| < r\delta_\epsilon} |f\left(\frac{k}{r}\right) - f(x)| b_{k,r}(x) + \sum_{|k-rx| \geq r\delta_\epsilon} |f\left(\frac{k}{r}\right) - f(x)| b_{k,r}(x)$$

$$\leq \frac{\epsilon}{2} + 2\|f\|_\infty \Pr(|S_{r,x} - rx| \geq r\delta_\epsilon)$$

$$\leq \frac{\epsilon}{2} + 2\|f\|_\infty \frac{x(1 - x)}{r^2 \delta_\epsilon^2} \quad \text{(by Chebyshev’s inequality)}$$

$$\leq \frac{\epsilon}{2} + \frac{\|f\|_\infty}{2r\delta_\epsilon^2}$$

$$\leq \epsilon$$

where the final inequality uses the assumption on $r$.  

\[\square\]