Weierstrass approximation theorem

COMS 4995-1 Spring 2020 (Daniel Hsu)

Theorem (Weierstrass approximation theorem). Suppose $f\colon [0,1] \to \mathbb{R}$ is continuous. For any $\epsilon > 0$, there exists a polynomial p such that

$$\sup_{x \in [0,1]} |f(x) - p(x)| \leq \epsilon.$$

Proof. Since f is continuous on $[0,1]$, it is *uniformly continuous*. This means that for any $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that $|f(x) - f(y)| < \epsilon/2$ for all $x, y \in [0,1]$ satisfying $|x - y| < \delta_\epsilon$. Let us fix an $\epsilon > 0$ and such a corresponding $\delta_\epsilon > 0$.

Let r be any positive integer such that $r \geq \frac{\|f\|_\infty}{\delta_\epsilon^2 \epsilon}$. Define the *Bernstein polynomials*

$$b_{k,r}(x) = \Pr(S_{r,x} = k) = \binom{r}{k} x^k (1-x)^{r-k}$$

where $S_{r,x} \sim \mathrm{Binom}(r, x)$. Let $p(x) := \sum_{k=0}^r f(\frac{k}{r}) b_{k,r}(x)$, which is a degree-$r$ polynomial. Then, for any $x \in [0,1]$,

$$\begin{aligned}
&|p(x) - f(x)| \\
&= \left| \sum_{k=0}^r \left(f(\tfrac{k}{r}) - f(x) \right) b_{k,r}(x) \right| \\
&\leq \sum_{|k - rx| < r\delta_\epsilon} |f(\tfrac{k}{r}) - f(x)| b_{k,r}(x) + \sum_{|k - rx| \geq r\delta_\epsilon} |f(\tfrac{k}{r}) - f(x)| b_{k,r}(x) \\
&\leq \frac{\epsilon}{2} + 2\|f\|_\infty \Pr(|S_{r,x} - rx| \geq r\delta_\epsilon) \\
&\leq \frac{\epsilon}{2} + 2\|f\|_\infty \frac{x(1-x)}{r\delta_\epsilon^2} \quad \text{(by Chebyshev's inequality)} \\
&\leq \frac{\epsilon}{2} + \frac{\|f\|_\infty}{2r\delta_\epsilon^2} \\
&\leq \epsilon
\end{aligned}$$

where the final inequality uses the assumption on r. ∎