Haussler's proof of Sauer's lemma

COMS 4995-1 Spring 2020 (Daniel Hsu)

Let $H \subseteq \{0,1\}^n$. We can identify H with a family of subsets of $[n] := \{1,\ldots,n\}$. A set $S \subseteq [n]$ is *shattered* by H if for every subset $T \subseteq S$, there exists $A \in H$ such that $A \cap S = T$. We say H has VC dimension d if the largest set shattered by H has cardinality d.

Sauer's lemma gives the following bound on the cardinality of H:

$$|H| \leq \binom{n}{\leq d} := \sum_{k=0}^{d} \binom{n}{k}.$$

Haussler's proof of Sauer's lemma is as follows. We shall repeatedly apply operations to H to obtain a different family G of subsets of $[n]$ with the following properties.

1. $|G| = |H|$.
2. If S is in G, then every subset of S is also in G.
3. If S is shattered by G, then it is also shattered by H.

By Property 1, it suffices to bound the cardinality of G. By Property 2, every $S \in G$ is shattered by G. Therefore, with Property 3, it follows that every $S \in G$ is shattered by H. Since H has VC dimension at most d, every $S \in G$ has cardinality at most d. So $|G| \leq \binom{n}{\leq d}$.

We now show how to construct this family G that satisfies these three properties. The construction is provided by the following algorithm.

- Initialize $G := H$.
- Loop until there there is no change to G:
  - Execute $\mathsf{SHIFT}(G,i)$ for all $i = 1,\ldots,n$.

This algorithm depends on the subroutine $\mathsf{SHIFT}(G,i)$, given as follows.

- $\mathsf{SHIFT}(G,i)$:
  - For each $A \in G$:
    - If $A \setminus \{i\} \notin G$, then replace A with $A \setminus \{i\}$ in G.

The top-level algorithm clearly terminates, since every step removes an item from one of the sets in G. Moreover, it does not change the cardinality of G, so Property 1 holds. Upon termination, Property 2 holds as well, since if there is a set $S \in G$ such that $T \notin G$ for some (maximal) $T \subsetneq S$, then we can run $\mathsf{SHIFT}(G, i)$ for some $i \in S \setminus T$ to change G.

It remains to argue that Property 3 holds. We show that if S is shattered by G after $\mathsf{SHIFT}(G, i)$, then it must have been shattered by G before $\mathsf{SHIFT}(G, i)$. Let G^0 be the family G before $\mathsf{SHIFT}(G, i)$, and let G^1 be the family G after $\mathsf{SHIFT}(G, i)$. Clearly if S does not contain i, then its shatteredness is not affected by $\mathsf{SHIFT}(G, i)$. So let us assume $i \in S$, and that it is shattered by G^1. Fix any $T \subseteq S$. If $i \in T$, then since S is shattered by G^1, there exists $A^1 \in G^1$ such that $A^1 \cap S = T \ni i$. In particular, $i \in A^1$. But the only new sets in G^1 are sets that do not contain i. So it must be that $A^1 \in G^0$. If $i \notin T$, then since S is shattered by G^1, it follows that there exists $A^1 \in G^1$ such that $A^1 \cap S = T \cup \{i\}$. In particular, $(A^1 \setminus \{i\}) \cap S = T$. But $A^1 \setminus \{i\} \in G^0$, since otherwise A^1 would have been shifted. In either case, there is a set $A^0 \in G^0$ such that $A^0 \cap S = T$. Since this holds for every $T \subseteq S$, it follows that S is shattered by G^0. This proves that Property 3 holds.