Moment bound for sums of random variables

COMS 4995-1 Spring 2020 (Daniel Hsu)

This moment bound is due to Ravi Kannan (who says it may be folklore).

**Theorem.** Let $X_1, \ldots, X_n$ be mean-zero random variables such that, for some $v_1, \ldots, v_n \geq 0$ and some $q \in \mathbb{N}$,

$$|\mathbb{E}X_i^p| \leq \frac{v_i}{2^p}, \quad p = 2, \ldots, q, \quad i = 1, \ldots, n.$$

Let $S := X_1 + \cdots + X_n$ and $v := v_1 + \cdots + v_n$. Pick any even $k$ such that $2 \leq k \leq \min\{q, v/4\}$, and assume $X_1, \ldots, X_n$ are $k$-wise independent. Then

$$\mathbb{E}S^k \leq v^{k/2} \cdot \frac{k!}{(k/2)!}.$$

**Remark 1.** The moment condition required here is weaker than the one from Bernstein’s inequality (which puts the absolute values inside the expectations and also requires the moment bounds for all integers $p \geq 2$).

**Proof.** A multinomial expansion gives

$$(X_1 + \cdots + X_n)^k = \sum_{\vec{d}} \binom{k}{\vec{d}} \prod_{i=1}^n X_i^{d_i},$$

where the summation is over $\vec{d} = (d_1, \ldots, d_n) \in (\mathbb{N} \cup \{0\})^n$ such that $\sum_{i=1}^n d_i = k$, and $\binom{k}{\vec{d}} = \frac{k!}{d_1! \cdots d_n!}$ is the multinomial coefficient.

We first obtain a basic bound using the assumptions on the random variables. By linearity of expectation and $k$-wise independence,

$$M_k := \mathbb{E}(X_1 + \cdots + X_n)^k = \sum_{\vec{d}} \binom{k}{\vec{d}} \prod_{i=1}^n \mathbb{E}X_i^{d_i}.$$

Because $\mathbb{E}X_i = 0$ for all $i \in [n] := \{1, \ldots, n\}$, it follows that every non-zero term in the summation corresponds to a $\vec{d}$ such that $d_i = 0$ or $d_i \geq 2$ for all $i$. We henceforth only ever consider such $\vec{d}$. Fix such a $\vec{d}$, and let

$$T(\vec{d}) := \{i \in [n] : d_i \geq 2\}$$

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be its support. Then, the term corresponding to \( \vec{d} \) can be bounded using the moment condition as follows:

\[
\left( \frac{k}{\vec{d}} \right) \prod_{i=1}^{n} \mathbb{E} X_i^{d_i} = \left( \frac{k}{\vec{d}} \right) \prod_{i \in T(\vec{d})} \mathbb{E} X_i^{d_i} \leq \left( \frac{k}{\vec{d}} \right) \prod_{i \in T(\vec{d})} |\mathbb{E} X_i^{d_i}| \leq \left( \frac{k}{\vec{d}} \right) \prod_{i \in T(\vec{d})} \frac{v_i}{2} d_i = k! \prod_{i \in T(\vec{d})} \frac{v_i}{2}.
\]

Therefore

\[
M_k \leq \sum_{\vec{d}} k! \prod_{i \in T(\vec{d})} \frac{v_i}{2} = k! \sum_{\vec{d}} \prod_{i \in T(\vec{d})} \frac{v_i}{2} = : M_k'.
\]

We further bound \( M_k' \) using combinatorial arguments. We enumerate the \( \vec{d} \) by their support sets \( T(\vec{d}) \). Each support set \( T(\vec{d}) \) can have cardinality at most \( \frac{k}{2} \), since the \( d_i \) sum to \( k \) and every non-zero \( d_i \) is at least two. So, we can write the summation over \( \vec{d} \) as

\[
k! \sum_{\vec{d}} \prod_{i \in T(\vec{d})} \frac{v_i}{2} = k! \sum_{t=1}^{k/2} \sum_{T \in \binom{[n]}{t}} \prod_{i \in T} \frac{v_i}{2} = k! \sum_{t=1}^{k/2} \sum_{T \in \binom{[n]}{t}} \left( \prod_{i \in T} \frac{v_i}{2} \right) |\{ \vec{d} : T(\vec{d}) = T \}|.
\]

Above, \( \binom{[n]}{t} \) denotes the family of all \( t \)-element subsets of \([n]\). Now fix \( t \in [k/2] \) and \( T \in \binom{[n]}{t} \). We first bound the number of \( \vec{d} \) with \( T(\vec{d}) = T \). Consider \( k \) balls and \( t \) bins (with the bins labeled by \( T \)). We want the number of allocations of balls to bins such that every bin has at least two balls. After allocating two balls to each bin, there are \( k - 2t \) balls remaining to allocate. The number of allocations is, by a stars-and-bars argument,

\[
\left( \frac{(k - 2t) + (t - 1)}{t - 1} \right) = \left( \frac{k - t - 1}{t - 1} \right).
\]

Therefore

\[
k! \sum_{t=1}^{k/2} \sum_{T \in \binom{[n]}{t}} \left( \prod_{i \in T} \frac{v_i}{2} \right) |\{ \vec{d} : T(\vec{d}) = T \}| = k! \sum_{t=1}^{k/2} \sum_{T \in \binom{[n]}{t}} \left( \prod_{i \in T} \frac{v_i}{2} \right) \left( \frac{k - t - 1}{t - 1} \right).
\]

Now we bound the rest of the summation over \( T \). The multinomial expansion of

\[
\left( \frac{v_1}{2} + \cdots + \frac{v_n}{2} \right)^t
\]

is
contains every term $\prod_{i \in T} (v_i/2)$ precisely $t!$ times; the remaining terms are non-negative. Therefore,

$$\sum_{T \in \binom{[n]}{t}} \prod_{i \in T} \frac{v_i}{2} \leq \frac{1}{t!} \left( \frac{v_1}{2} + \cdots + \frac{v_n}{2} \right)^t = \frac{v^t}{2^{t!}}.$$ 

Combining this with the previous bound gives

$$M'_k \leq k! \sum_{t=1}^{k/2} \sum_{T \in \binom{[n]}{t}} \left( \prod_{i \in T} \frac{v_i}{2} \right) \binom{k-t-1}{t-1} \leq k! \sum_{t=1}^{k/2} \frac{v^t}{2^{t!}} \binom{k-t-1}{t-1} =: M''_k.$$ 

Finally, we derive a simpler bound on $M''_k$. The binomial coefficient can be bounded using

$$\binom{k-t-1}{t-1} \leq 2^{k-t-1},$$ 

since the number of subsets of size $t-1$ is at most the number of subsets overall. Therefore,

$$M''_k \leq 2^{k-1} k! \sum_{t=1}^{k/2} \left( \frac{v}{4} \right)^t \frac{1}{t!}. $$

Since $v/4 \geq k \geq 2t$ by assumption on $k$, the largest term in the summation is one corresponding to $t = k/2$, and the $t$-th term is at least twice the $(t - 1)$-th term. It follows that

$$\sum_{t=1}^{k/2} \left( \frac{v}{4} \right)^t \frac{1}{t!} \leq \left( \frac{v}{4} \right)^{k/2} \frac{1}{(k/2)!} \left( 1 + \frac{1}{2} + \frac{1}{4} + \cdots \right) \leq 2 \left( \frac{v}{4} \right)^{k/2} \frac{1}{(k/2)!}.$$ 

Therefore,

$$M''_k \leq 2^{k-1} k! \sum_{t=1}^{k/2} \left( \frac{v}{4} \right)^t \frac{1}{t!} \leq v^{k/2} \cdot \frac{k!}{(k/2)!}.$$ 

So, we conclude with the moment bound

$$\mathbb{E}S^k \leq v^{k/2} \cdot \frac{k!}{(k/2)!}.$$ 

\[ \square \]