Moment bound for sums of random variables

COMS 4995-1 Spring 2020 (Daniel Hsu)

This moment bound is due to Ravi Kannan (who says it may be folklore).

Theorem. Let $X_1, \dots, X_n$ be mean-zero random variables such that, for some $v_1, \dots, v_n \geq 0$ and some $q \in \mathbb{N}$,

$$|\mathbb{E}X_i^p| \leq \frac{v_i}{2} p!, \quad p = 2, \dots, q, \quad i = 1, \dots, n.$$

Let $S := X_1 + \cdots + X_n$ and $v := v_1 + \cdots + v_n$. Pick any even k such that $2 \leq k \leq \min\{q, v/4\}$, and assume $X_1, \dots, X_n$ are k-wise independent. Then

$$\mathbb{E}S^k \leq v^{k/2} \cdot \frac{k!}{(k/2)!}.$$

Remark 1. The moment condition required here is weaker than the one from Bernstein's inequality (which puts the absolute values inside the expectations and also requires the moment bounds for all integers $p \geq 2$).

Proof. A multinomial expansion gives

$$(X_1 + \cdots + X_n)^k = \sum_{\vec{d}} \binom{k}{\vec{d}} \prod_{i=1}^n X_i^{d_i},$$

where the summation is over $\vec{d} = (d_1, \dots, d_n) \in (\mathbb{N} \cup \{0\})^n$ such that $\sum_{i=1}^n d_i = k$, and $\binom{k}{\vec{d}} = \frac{k!}{d_1! \cdots d_n!}$ is the multinomial coefficient.

We first obtain a basic bound using the assumptions on the random variables. By linearity of expectation and k-wise independence,

$$M_k := \mathbb{E}(X_1 + \cdots + X_n)^k = \sum_{\vec{d}} \binom{k}{\vec{d}} \prod_{i=1}^n \mathbb{E}X_i^{d_i}.$$

Because $\mathbb{E}X_i = 0$ for all $i \in [n] := \{1, \dots, n\}$, it follows that every non-zero term in the summation corresponds to a $\vec{d}$ such that $d_i = 0$ or $d_i \geq 2$ for all i. We henceforth only ever consider such $\vec{d}$. Fix such a $\vec{d}$, and let

$$T(\vec{d}) := \{i \in [n] : d_i \geq 2\}$$

be its support. Then, the term corresponding to $\vec{d}$ can be bounded using the moment condition as follows:

$$\binom{k}{\vec{d}}\prod_{i=1}^{n}\mathbb{E}X_i^{d_i} = \binom{k}{\vec{d}}\prod_{i\in T(\vec{d})}\mathbb{E}X_i^{d_i} \le \binom{k}{\vec{d}}\prod_{i\in T(\vec{d})}|\mathbb{E}X_i^{d_i}| \le \binom{k}{\vec{d}}\prod_{i\in T(\vec{d})}\frac{v_i}{2}d_i! = k!\prod_{i\in T(\vec{d})}\frac{v_i}{2}.$$

Therefore

$$M_k \le \sum_{\vec{d}} k!\prod_{i\in T(\vec{d})}\frac{v_i}{2} = k!\sum_{\vec{d}}\prod_{i\in T(\vec{d})}\frac{v_i}{2} =: M_k'.$$

We further bound M_k' using combinatorial arguments. We enumerate the $\vec{d}$ by their support sets $T(\vec{d})$. Each support set $T(\vec{d})$ can have cardinality at most $k/2$, since the d_i sum to k and every non-zero d_i is at least two. So, we can write the summation over $\vec{d}$ as

$$k!\sum_{\vec{d}}\prod_{i\in T(\vec{d})}\frac{v_i}{2} = k!\sum_{t=1}^{k/2}\sum_{T\in\binom{[n]}{t}}\sum_{\substack{\vec{d}:\\ T(\vec{d})=T}}\prod_{i\in T}\frac{v_i}{2} = k!\sum_{t=1}^{k/2}\sum_{T\in\binom{[n]}{t}}\left(\prod_{i\in T}\frac{v_i}{2}\right)|\{\vec{d}:T(\vec{d})=T\}|.$$

Above, $\binom{[n]}{t}$ denotes the family of all t-element subsets of $[n]$. Now fix $t\in[k/2]$ and $T\in\binom{[n]}{t}$. We first bound the number of $\vec{d}$ with $T(\vec{d})=T$. Consider k balls and t bins (with the bins labeled by T). We want the number of allocations of balls to bins such that every bin has at least two balls. After allocating two balls to each bin, there are $k-2t$ balls remaining to allocate. The number of allocations is, by a stars-and-bars argument,

$$\binom{(k-2t)+(t-1)}{t-1} = \binom{k-t-1}{t-1}.$$

Therefore

$$k!\sum_{t=1}^{k/2}\sum_{T\in\binom{[n]}{t}}\left(\prod_{i\in T}\frac{v_i}{2}\right)|\{\vec{d}:T(\vec{d})=T\}| = k!\sum_{t=1}^{k/2}\sum_{T\in\binom{[n]}{t}}\left(\prod_{i\in T}\frac{v_i}{2}\right)\binom{k-t-1}{t-1}.$$

Now we bound the rest of the summation over T. The multinomial expansion of

$$\left(\frac{v_1}{2}+\cdots+\frac{v_n}{2}\right)^t$$

contains every term $\prod_{i\in T}(v_i/2)$ precisely $t!$ times; the remaining terms are non-negative. Therefore,

$$\sum_{T\in\binom{[n]}{t}} \prod_{i\in T} \frac{v_i}{2} \le \frac{1}{t!}\left(\frac{v_1}{2}+\cdots+\frac{v_n}{2}\right)^t = \frac{v^t}{2^t t!}.$$

Combining this with the previous bound gives

$$M_k' \le k! \sum_{t=1}^{k/2} \sum_{T\in\binom{[n]}{t}} \left(\prod_{i\in T}\frac{v_i}{2}\right)\binom{k-t-1}{t-1} \le k! \sum_{t=1}^{k/2} \frac{v^t}{2^t t!}\binom{k-t-1}{t-1} =: M_k''.$$

Finally, we derive a simpler bound on M_k''. The binomial coefficient can be bounded using

$$\binom{k-t-1}{t-1} \le 2^{k-t-1},$$

since the number of subsets of size $t-1$ is at most the number of subsets overall. Therefore,

$$M_k'' \le 2^{k-1} k! \sum_{t=1}^{k/2} \left(\frac{v}{4}\right)^t \frac{1}{t!}.$$

Since $v/4 \ge k \ge 2t$ by assumption on k, the largest term in the summation is one corresponding to $t = k/2$, and the t-th term is at least twice the $(t-1)$-th term. It follows that

$$\sum_{t=1}^{k/2} \left(\frac{v}{4}\right)^t \frac{1}{t!} \le \left(\frac{v}{4}\right)^{k/2} \frac{1}{(k/2)!}\left(1+\frac{1}{2}+\frac{1}{4}+\cdots\right) \le 2\left(\frac{v}{4}\right)^{k/2} \frac{1}{(k/2)!}.$$

Therefore,

$$M_k'' \le 2^{k-1} k! \sum_{t=1}^{k/2} \left(\frac{v}{4}\right)^t \frac{1}{t!} \le v^{k/2} \cdot \frac{k!}{(k/2)!}.$$

So, we conclude with the moment bound

$$\mathbb{E} S^k \le v^{k/2} \cdot \frac{k!}{(k/2)!}.$$

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