COMS 4773 Spring 2024 Homework 0

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This assignment is not to be turned in.

Problem 1. Let X be a non-negative integer-valued random variable.

(a) Prove that $\Pr(X \neq 0) \leq \mathbb{E} X$.

(b) Prove that $\Pr(X = 0) \leq \operatorname{var}(X) / \mathbb{E}(X^2)$.

Problem 2. For $p \geq 1$, the l^p norm for vectors in $\mathbb{R}^n$ is defined by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \quad \text{for all } x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

For $p = \infty$, the l^∞ norm is defined by

$$\|x\|_\infty = \max_{i=1,\ldots,n} |x_i| \quad \text{for all } x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

(a) Prove that for any vectors $x, y \in \mathbb{R}^n$,

$$x \cdot y \leq \|x\|_1 \|y\|_\infty.$$

(b) Prove that $\|x\|_2 \leq \|x\|_1 \leq \sqrt{n} \|x\|_2$ for all $x \in \mathbb{R}^n$.

Problem 3. Prove that for any vectors $v_1, \ldots, v_d \in \mathbb{R}^n$, each with $\|v_i\|_2 \leq 1$, any positive integer k, and any vector u in the convex hull of $v_1, \ldots, v_d$, there exists non-negative integers $k_1, \ldots, k_d$ with $\sum_{i=1}^{d} k_i = k$ such that

$$\left\| u - \frac{1}{k} \sum_{i=1}^{d} k_i v_i \right\|_2^2 \leq \frac{1}{k}.$$

Note: A vector u is in the convex hull of $v_1, \ldots, v_d$ if there exists non-negative numbers $\alpha_1, \ldots, \alpha_d \geq 0$ with $\sum_{i=1}^{d} \alpha_i = 1$ such that

$$u = \sum_{i=1}^{d} \alpha_i v_i.$$