# Nearest neighbors

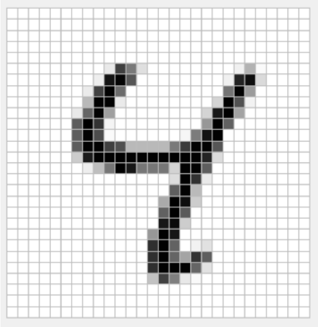
COMS 4771 Fall 2023

# Digit recognition

Problem: Create a program that, given an image of a handwritten digit as input, returns the digit depicted in the image

**Simplifying assumptions:**

- The image depicts some digit (from $\{0, 1, \ldots, 9\}$)
- The depicted digit is (roughly) in the center of the image
- The image is a $28 \times 28$ pixel image (for a total of $784$ pixels)
- Each pixel is grayscale; pixel intensity is an integer from $\{0, 1, \ldots, 255\}$



**Machine learning approach to digit recognition:**

- Don't explicitly write the image classifier by hand
- Collect a labeled dataset of images
  - Each image is an example of how someone might write a digit
  - Each image is annotated with a label—the digit depicted in the image
  - NIST has collected such a dataset with 60000 examples ("MNIST")[1]
- Provide the labeled dataset as input to a learning algorithm
- Learning algorithm returns an image classifier

[1] http://yann.lecun.com/exdb/mnist/

# Nearest neighbors learning algorithm

**Nearest Neighbors (NN) learning algorithm:**

- Input: Labeled dataset $\mathcal{S}$
- Output: NN classifier for labeled dataset $\mathcal{S}$ (also a program!)

Notation:

- $n$: number of images in the dataset
- $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$: the $n$ images
- $y^{(1)}, y^{(2)}, \ldots, y^{(n)}$: the $n$ corresponding labels
- Labeled dataset

$$\mathcal{S} = ((x^{(i)}, y^{(i)}))_{i=1}^{n} = ((x^{(1)}, y^{(1)}), (x^{(2)}, y^{(2)}), \ldots, (x^{(n)}, y^{(n)}))$$

- (Sometimes $x$'s and $y$'s come separately: $(x^{(i)})_{i=1}^{n}$ and $(y^{(i)})_{i=1}^{n}$)

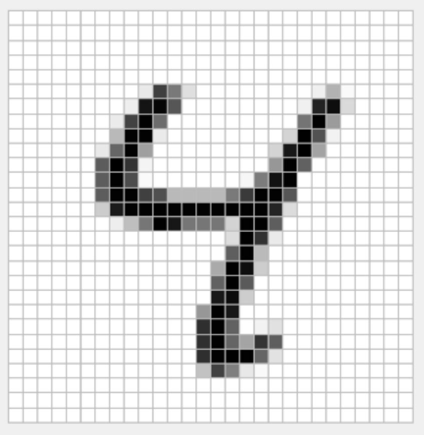
**NN classifier for labeled dataset $\mathcal{S}$:**

- Input: $x$
- Output: prediction of correct label of $x$
- Pseudocode:

# Euclidean distance

$$D(x, z) = \|x - z\|$$

# Image of digit as $784$-vector: pixel intensities as features



## Computational requirements of NN classifier:

- Memory

- Time

```python
import numpy as np

def learn(train_x, train_y):
  return (train_x, train_y)

def predict(params, test_x):
  x, y = params
  return y[np.argmin(np.sum(x**2, axis=1) - 2*test_x.dot(x.T),
   ↪  axis=1)]
```

If you want to strictly follow the idea that “learn” should return a function:

```python
def learn(train_x, train_y):
  return lambda test_x: train_y[np.argmin(np.sum(train_x**2, axis=1)
   ↪  - 2*test_x.dot(train_x.T), axis=1)]
```

# Evaluating a classifier

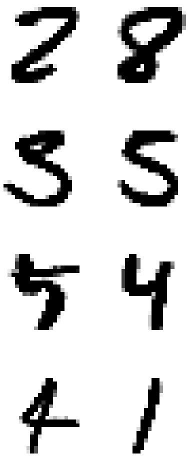
- Error rate on classifier $f$ on labeled dataset:

- Training error rate (i.e., error rate on $\mathcal{S}$) of NN classifier:

NIST has provided **separate collection of 10000 labeled examples**, which we **did not provide to NN learning algorithm**

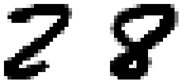
- We use it as test data
- Test error rate (i.e., error rate on test data) of NN classifier:

Test image, nearest neighbor in training data:

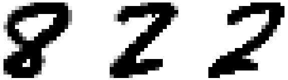


# Upgrading NN: more neighbors

Test image, nearest neighbor in training data:



$3$ closest images in training data:



## $k$-NN classifier for labeled dataset $\mathcal{S}$:

- Input: $x$
- Output: prediction of correct label of $x$
- Pseudocode:

| hyperparameter $k$ | 1 | 3 | 5 | 7 | 9 |
|---|---|---|---|---|---|
| test error rate | 3.09% | 2.95% | 3.12% | 3.06% | 3.41% |

Hyperparameter tuning (e.g., how to choose $k$?)

- Cross validation: use subset of training data to act as test data for purpose of evaluating different hyperparameter choices
- Pseudocode:

# Upgrading NN: better distances

## Other types of distances

- $\ell^p$ distance for $d$-vectors $x = (x_1, \ldots, x_d)$

$$D_p(x, z) = \left(|x_1 - z_1|^p + \cdots + |x_d - z_d|^p\right)^{1/p}$$

## Other types of distances

- "Edit distance" for strings (e.g., $x =$ "kitten")

$$D_{\text{edit}}(x, z) = \#\ \text{insertions/deletions/swaps needed to transform } x \text{ to } z$$

Digit recognition using NN classifier based on different distances

| distance metric | $\ell^2$ | $\ell^3$ | "shape" |
|---|---|---|---|
| test error rate | 3.09% | 2.83% | $< 1\%$ |

Caution: many types of distances (e.g., $\ell^p$ distances) are sensitive to the quality of the numerical features

- 1000 "noisy" pixels with random intensity values

- Single "noisy" pixel with scale 1000 times that of regular pixels

**"Curse of dimension":** weird effects in "high dimensional" feature spaces (e.g., space of all $d$-vectors for large $d$)

Question: How can we choose the distance function to use?