Optimization by gradient methods

COMS 4771 Fall 2023

Unconstrained optimization problems

Common form of optimization problem in machine learning:

$$\min_{w \in \mathbb{R}^d} \quad J(w)$$

We would like an algorithm that, given the objective function J, finds particular setting of w so that $J(w)$ is as small as possible

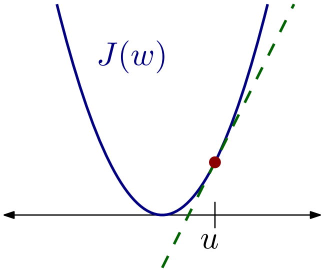
- What does it mean to be "given J"?
- What types of objective functions can we hope to minimize?

Review of multivariate differential calculus

A function $J\colon \mathbb{R}^d \to \mathbb{R}$ is differentiable if, for every $u \in \mathbb{R}^d$, there is an affine function $A\colon \mathbb{R}^d \to \mathbb{R}$ such that

$$\lim_{w \to u} \frac{J(w) - A(w)}{\|w - u\|} = 0$$

Affine function A is called the (best) affine approximation of J at u



A may depend on u—i.e., possibly a different A for each u

About the affine approximation:

- Since A is affine, we can write it as

$$A(w) = \underline{\hspace{4cm}}$$

- $m \in \mathbb{R}^d$ is the "slope" (and specifies a linear function)
- $b \in \mathbb{R}$ is the "intercept"
- The intercept must be $b = \underline{\hspace{5cm}}$ because

$$J(u) = \underline{\hspace{6cm}}$$

- So we can write A as

$$A(w) = J(u) + m^\intercal (w - u)$$

About the affine approximation:
Letting $e^{(1)}, \ldots, e^{(d)}$ be standard coordinate basis for $\mathbb{R}^d$, write $m = \sum_{i=1}^d m_i \, e^{(i)}$

Since $A(w) = J(u) + m^\intercal (w - u)$ is best affine approximation of J at u,

$$0 = \lim_{t \to 0} \frac{J(u + te^{(i)}) - A(u + te^{(i)})}{|t|} = \lim_{t \to 0} \frac{J(u + te^{(i)}) - (J(u) + tm_i)}{|t|}$$

since $u + te^{(i)}$ differs from u by $t \in \mathbb{R}$ in the i-th coordinate

Whether t approaches zero from left or right, we find

$$m_i = \lim_{t \to 0} \underline{\hspace{6cm}} = \underline{\hspace{3cm}}$$

Vector-valued function (a.k.a. vector field) of all partial derivatives of J is called the gradient of J, written $\nabla J \colon \mathbb{R}^d \to \mathbb{R}^d$

$$\nabla J(u) = \left(\frac{\partial J}{\partial w_1}(u), \ldots, \frac{\partial J}{\partial w_d}(u) \right)$$

Summary: If $J\colon \mathbb{R}^d \to \mathbb{R}$ is differentiable, then for any $u \in \mathbb{R}^d$,

$$\lim_{w \to u} \frac{J(w) - (J(u) + \nabla J(u)^\mathsf{T}(w - u))}{\|w - u\|} = 0$$

Gradient descent

(Back to $\min_{w \in \mathbb{R}^d} J(w)$ where J is differentiable)

Question: Given candidate setting of variables $w = u \in \mathbb{R}^d$, achieving objective value $J(u)$, how can we change u to achieve a lower objective value?

Upshot: Modify u by subtracting $\eta \nabla J(u)$ for some $\eta > 0$

Caveat: Approximations in our argument are OK only if "change" is "small enough" (which means η should be "small enough")

Gradient descent: iterative method that attempts to minimize $J\colon \mathbb{R}^d \to \mathbb{R}$

- Initialize $w^{(0)} \in \mathbb{R}^d$
- For iteration $t = 1, 2, \ldots$ until "stopping condition" is satisfied:

$$w^{(t)} \leftarrow w^{(t-1)} - \eta_t \nabla J(w^{(t-1)}) \qquad \text{(update rule)}$$

- Return final $w^{(t)}$

What's missing in this algorithm description?

Examples of gradient descent algorithms

Sum of squared errors objective from OLS

$$J(w) = \sum_{(x,y)\in\mathcal{S}} (x^{\mathsf{T}}w - y)^2$$

for dataset $\mathcal{S}$ from $\mathbb{R}^d \times \mathbb{R}$

- Use linearity and chain rule to get formula for $\frac{\partial J}{\partial w_i}$:

$$\frac{\partial J}{\partial w_i}(w) = \sum_{(x,y)\in\mathcal{S}} \underline{\hspace{6cm}}$$

- Therefore

$$\nabla J(w) = \sum_{(x,y)\in\mathcal{S}} \underline{\hspace{6cm}}$$

- Update rule in iteration t:

$$w^{(t)} \leftarrow w^{(t-1)} - \eta_t \sum_{(x,y)\in\mathcal{S}} \underline{\hspace{7cm}}$$

Negative log-likelihood from logistic regression

$$J(w) = \sum_{(x,y) \in \mathcal{S}} \left(\ln(1 + e^{x^\mathsf{T} w}) - y x^\mathsf{T} w \right)$$

for dataset $\mathcal{S}$ from $\mathbb{R}^d \times \{0, 1\}$

- Use linearity and chain rule to get formula for $\frac{\partial J}{\partial w_i}$:

$$\frac{\partial J}{\partial w_i}(w) = \sum_{(x,y) \in \mathcal{S}} \underline{\hspace{6cm}}$$

- Therefore

$$\nabla J(w) = \sum_{(x,y) \in \mathcal{S}} \underline{\hspace{6cm}}$$

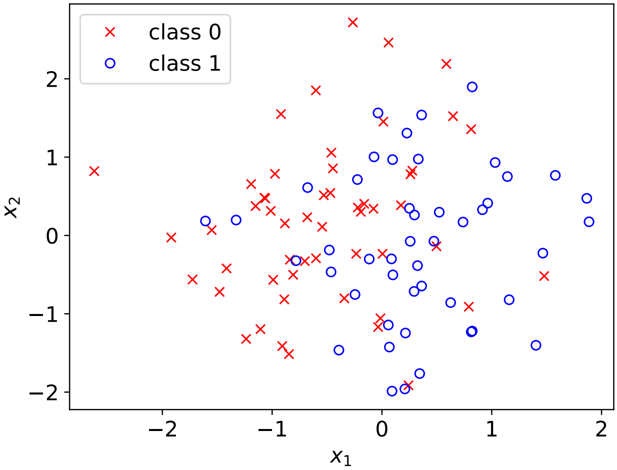
- Update rule in iteration t:

$$w^{(t)} \leftarrow w^{(t-1)} - \eta_t \sum_{(x,y) \in \mathcal{S}} \underline{\hspace{7cm}}$$

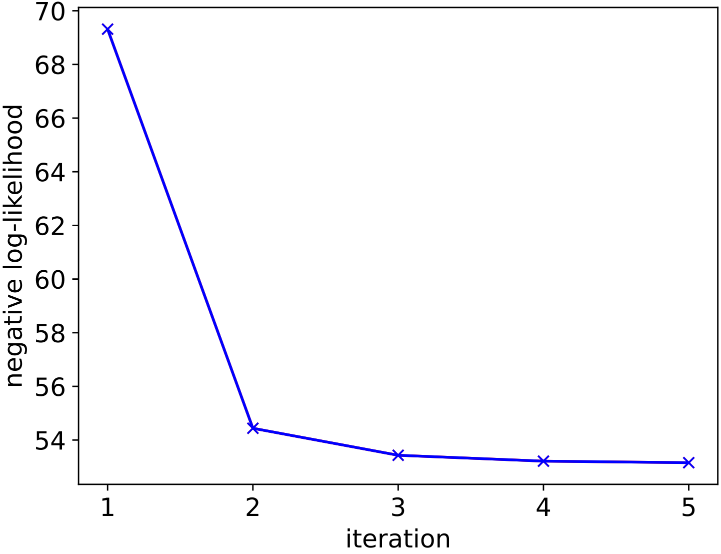
```python
def learn(train_x, train_y, eta=0.1, num_steps=1000):
  w = np.zeros(train_x.shape[1])
  for t in range(num_steps):
    w += eta * (train_y - 1/(1+np.exp(-train_x.dot(w)))).dot(train_x)
  return w
```

Synthetic example: $X \sim \mathrm{N}((0,0), I)$, conditional distribution of Y given $X = x$ is $\mathrm{Bernoulli}(\mathrm{logistic}(w^{\mathsf{T}}x))$ for $w = (3/2, -1/2)$

- $n = 100$ training examples $\mathcal{S} \overset{\text{i.i.d.}}{\sim} (X, Y)$



$\eta_t = 0.1$ starting from $w^{(0)} = (0, 0)$



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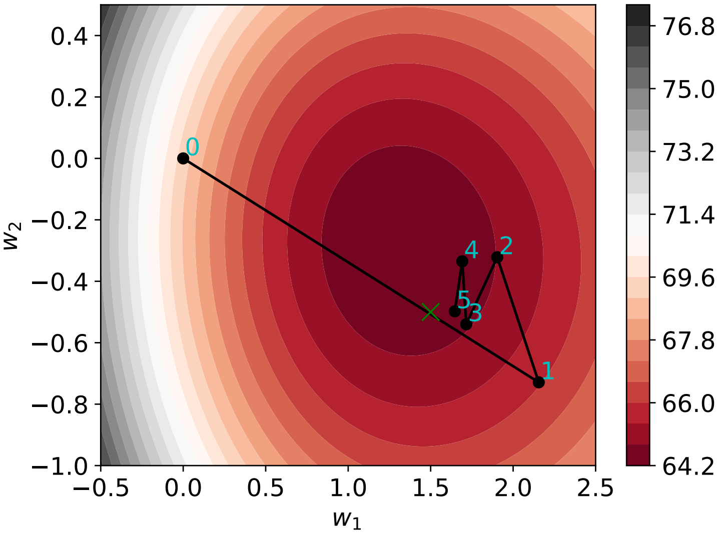
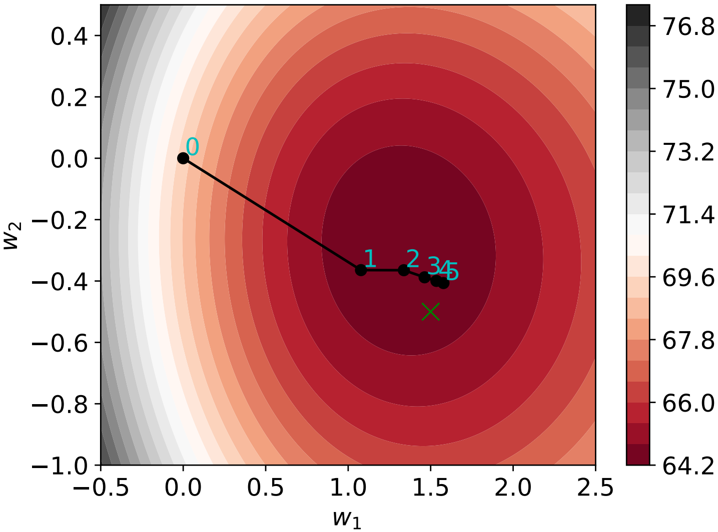
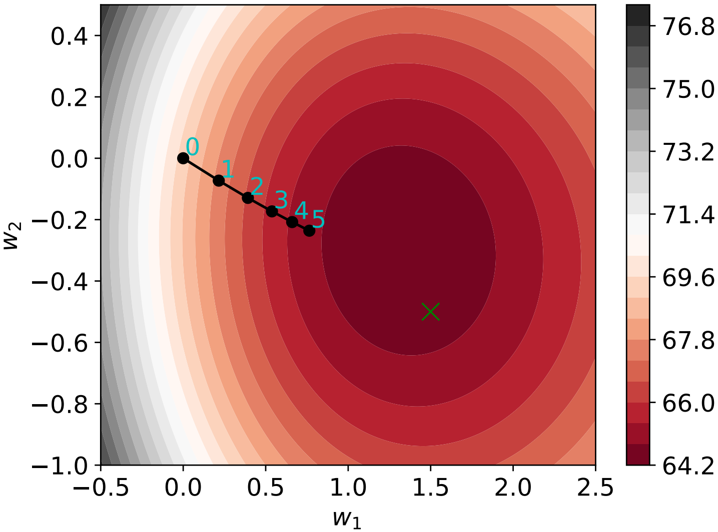


$\eta_t = 0.05$ starting from $w^{(0)} = (0, 0)$



$\eta_t = 0.01$ starting from $w^{(0)} = (0, 0)$



Guarantees about gradient descent

Guarantee about gradient descent updates: If J is "smooth enough", then there is a choice for $\eta > 0$ such that, for any $u \in \mathbb{R}^d$,

$$J(u - \eta \nabla J(u)) \leq J(u) - \frac{\eta}{2} \|\nabla J(u)\|^2$$

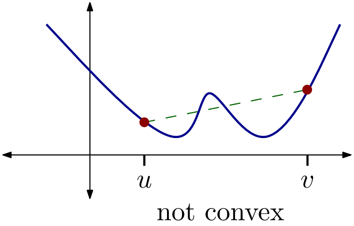
Guarantee about gradient descent for convex objectives: If J is convex and "smooth enough", then there is a choice for $\eta > 0$ such that, for any $w^{(0)} \in \mathbb{R}^d$, iterates of gradient descent $w^{(1)}, w^{(2)}, \ldots$ (with $\eta_t = \eta$) satisfy

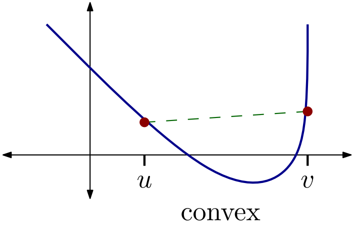
$$\lim_{t \to \infty} J(w^{(t)}) = \min_{w \in \mathbb{R}^d} J(w)$$

Convex functions

A function $J \colon \mathbb{R}^d \to \mathbb{R}$ is convex if, for all $u, v \in \mathbb{R}^d$, and all $\alpha \in [0, 1]$,

$$J((1-\alpha)u + \alpha v) \leq (1-\alpha)J(u) + \alpha J(v)$$

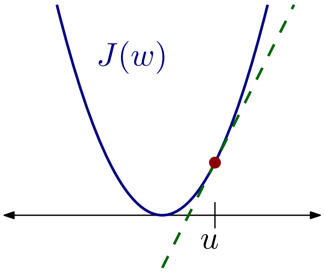




A differentiable function $J\colon \mathbb{R}^d \to \mathbb{R}$ is convex if, for all $u, w \in \mathbb{R}^d$,

$$J(w) \geq J(u) + \nabla J(u)^{\mathsf{T}}(w - u)$$

i.e., J lies above all of its affine approximations



A continuously twice-differentiable function $J\colon \mathbb{R}^d \to \mathbb{R}$ is convex if, for all $u \in \mathbb{R}^d$, the $d \times d$ matrix of second derivatives of J at u is positive semidefinite

Operations that preserve convexity:

- Sum of convex functions $J_1\colon \mathbb{R}^d \to \mathbb{R}$ and $J_2\colon \mathbb{R}^d \to \mathbb{R}$

$$J(w) = J_1(w) + J_2(w)$$

- Non-negative scalar multiple of a convex function $J_0\colon \mathbb{R}^d \to \mathbb{R}$

$$J(w) = c\, J_0(w), \quad c \geq 0$$

- Max of convex functions $J_1\colon \mathbb{R}^d \to \mathbb{R}$ and $J_2\colon \mathbb{R}^d \to \mathbb{R}$

$$J(w) = \max\{J_1(w), J_2(w)\}$$

- Composition of convex function $J_0\colon \mathbb{R}^k \to \mathbb{R}$ with affine mapping

$$J(w) = J_0(Mw + b)$$

  for $M \in \mathbb{R}^{k\times d}$ and $b \in \mathbb{R}^k$

Example: sum of squared errors $J(w) = \sum_{(x,y)\in\mathcal{S}}(x^\mathsf{T} w - y)^2$

Why convexity of J helps with gradient descent:

- Convexity ensures negative gradient $-\nabla J(u)$ satisfies

$$(-\nabla J(u))^{\mathsf{T}}(w-u) \geq J(u) - J(w)$$

  for all $u, w \in \mathbb{R}^d$
- Suppose w is minimizer of J, and you currently have u in hand
- Ideal direction to move in: $\delta = w - u$

Stochastic gradient descent

Many objective functions in machine learning are decomposable, i.e., can be written as sum

$$J(w) = \sum_{i=1}^{n} J^{(i)}(w)$$

E.g., sum of losses on training examples

$$J^{(i)}(w) = \text{loss}(f_w(x^{(i)}), y^{(i)})$$

Computational cost to compute $\nabla J(w)$?

Alternative: instead of using

$$\nabla J(w) = \sum_{i=1}^{n} \nabla J^{(i)}(w),$$

just use one of the terms in the sum (chosen uniformly at random)

Stochastic gradient descent (SGD) for $J(w) = \sum_{i=1}^{n} J^{(i)}(w)$

- Initialize $w^{(0)} \in \mathbb{R}^d$
- For iteration $t = 1, 2, \ldots$ until "stopping condition" is satisfied:

$$w^{(t)} \leftarrow w^{(t-1)} - \eta_t \nabla J^{(I_t)}(w^{(t-1)}) \quad \text{where } I_t \sim \mathrm{Unif}(\{1, \ldots, n\})$$

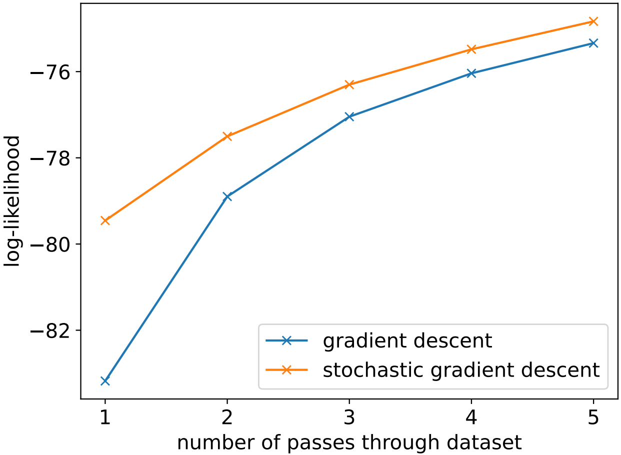
- Return final $w^{(t)}$

Some practical variants of SGD:

- Use sampling without replacement to choose $I_1, I_2, \ldots, I_n$ (i.e., go through terms in a uniformly random order)
  - Called SGD without replacement
- Instead of updating with gradient of single term, update with sum of gradients for next B terms
  - Called minibatch SGD; B is the minibatch size

Iris dataset, treating versicolor and virginica as a single class

- Maximizing log-likelihood in logistic regression with gradient descent and with SGD (both using $\eta_t = 0.01$, starting from $w^{(0)} = (0, 0)$)



Practical considerations

- Conditioning

- Initialization $w^{(0)} \in \mathbb{R}^d$

- Choice of "step size" $\eta_t > 0$ (a.k.a. "learning rate")

- Stopping condition

Automatic differentiation

Primary "technical work" in implementing gradient descent method:
Derive formula and write code for gradient computation ∇J

- Like doing long division by hand (i.e., without electronic calculators)
- Fairly straightforward, but can be tedious and easy to make mistakes

Automatic differentiation (autodiff):

- Method for automatically computing derivatives of functions specified by straight-line programs
- Gradient of a function can be computed this way in the roughly same amount of time it takes to compute the function itself (!)

Example: $J(w) = x^\intercal w$

- For each $j = 1, \ldots, d$, compute

$$\frac{\partial J}{\partial w_j}(w) = \_\_\_\_\_$$

- Time to compute function and gradient: ________

Example: $J(w) = g(f(w))$ where $f(w) = x^\intercal w$ and $g(t) = \text{logistic}(t)$

- For each $j = 1, \dots, d$, compute

$$\frac{\partial J}{\partial w_j}(w) = \underline{\hspace{6cm}}$$

- Time to compute function: ________
- Time to compute gradient: naïvely $O(d^2)$, but easy to get $O(d)$

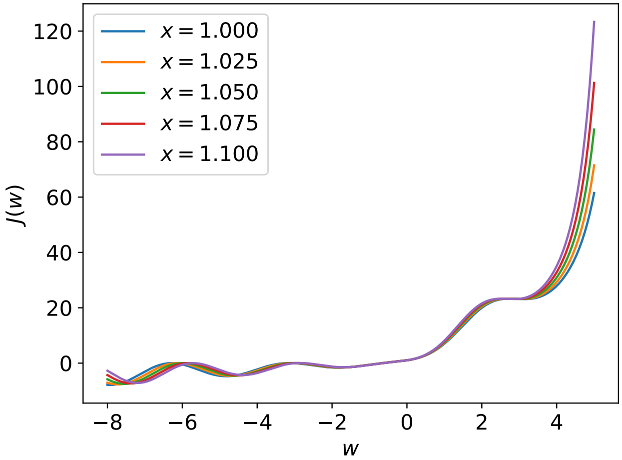
Example: tower of exponentials $J(w) = \exp(\exp(\exp(\cdots \exp(xw) \cdots)))$
(for scalar x and w)

We only want single number ($\frac{\partial J}{\partial w}$), but function is more complicated

$$\frac{\partial}{\partial w}\{e^{e^{e^{e^{e^{e^{e^{xw}}}}}}}\} = e^{e^{e^{e^{e^{e^{e^{xw}}}}}}} e^{e^{e^{e^{e^{e^{xw}}}}}} e^{e^{e^{e^{e^{xw}}}}} e^{e^{e^{e^{xw}}}} e^{e^{e^{xw}}} e^{e^{xw}} e^{xw} x$$

- Time to compute tower of exponentials of height h: ________
- Time to compute derivative: ______________________________

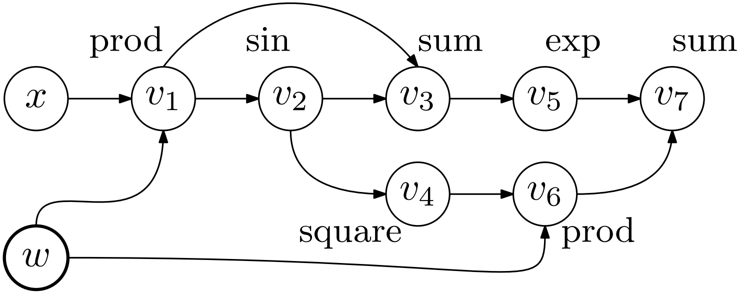
Example: $J(w) = \exp(xw + \sin(xw)) + \sin^2(xw)w$
(for scalar x and w)



Write as J as a straight-line program: each line declares a new variable as a function of inputs (e.g., w), constants (e.g., x), or previously defined variables

$$J(w) = \exp(xw + \sin(xw)) + \sin^2(xw)w$$

$$v_1 := \text{prod}(x, w)$$
$$v_2 := \sin(v_1)$$
$$v_3 := \text{sum}(v_1, v_2)$$
$$v_4 := \text{square}(v_2)$$
$$v_5 := \exp(v_3)$$
$$v_6 := \text{prod}(v_4, w)$$
$$v_7 := \text{sum}(v_5, v_6)$$



Computation directed acyclic graph $G = (V, E)$

All functions used in straight-line program must come with subroutines for computing "local" partial derivative

Example:

$$
\begin{aligned}
v_6 &:= \mathrm{prod}(v_4, w) \\
\frac{\partial v_6}{\partial v_4} &= \frac{\partial\, \mathrm{prod}(v_4, w)}{\partial v_4} = w \\
\frac{\partial v_6}{\partial w} &= \frac{\partial\, \mathrm{prod}(v_4, w)}{\partial w} = v_4
\end{aligned}
$$

Stage 1: **Forward pass**

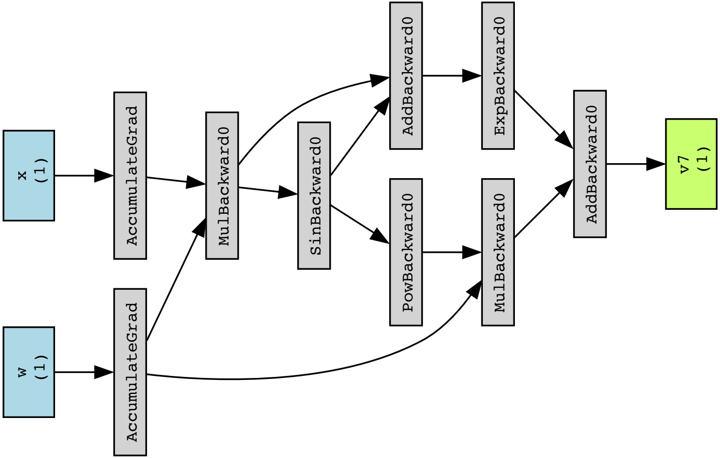
- Compute value of each node given inputs in a forward pass through the G (starting from inputs x and w)
- Save values at all intermediate nodes

Stage 2: **Backward pass**

- Compute partial derivative $\frac{\partial v_7}{\partial v}$ of output (v_7) with respect to each node variable v, evaluated at current node values
- Do this in reverse topological order; save intermediate results!

Chain rule: $$\frac{\partial v_7}{\partial v} = \sum_{(v,u)\in E} \frac{\partial v_7}{\partial u} \cdot \frac{\partial u}{\partial v}$$

- Time to compute function and partial derivatives: $O(|V| + |E|)$
- Modern numerical software facilitates construction of computation graph



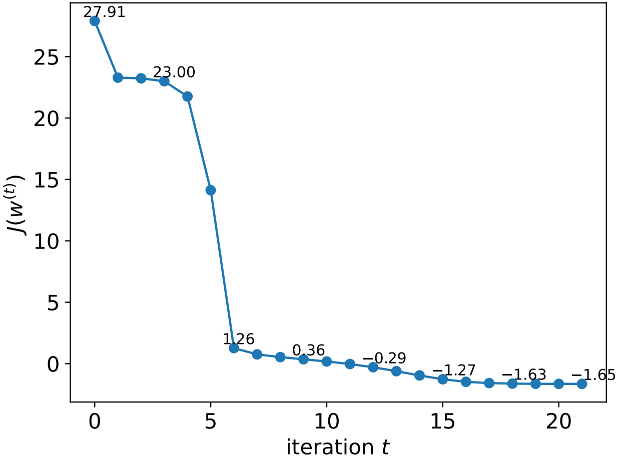
Setup

```python
import torch

x = torch.Tensor([1])
w = torch.Tensor([4])
w.requires_grad = True

def J(w):
  v1 = x * w
  v2 = torch.sin(v1)
  v3 = v1 + v2
  v4 = torch.pow(v2, 2)
  v5 = torch.exp(v3)
  v6 = v4 * w
  v7 = v5 + v6
  return v7
```

Gradient descent code

```python
for t in range(22):
  objective_value = J(w)
  objective_value.backward()
  with torch.no_grad():
    w -= 0.1 * w.grad
    w.grad.zero_()
```

Gradient descent on $J(w)$, starting from $w^{(0)} = 4$, using $\eta_t = 0.1$



Converges to $w = -1.847$, $J(w) = -1.649$, $\frac{\partial J}{\partial w}(w) = 0$