AdaBoost

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The algorithm
The input training data is \( \{(x_i, y_i)\}_{i=1}^n \) from \( \mathcal{X} \times \{-1, +1\} \).

- Initialize \( D_1(i) := \frac{1}{n} \) for each \( i = 1, \ldots, n \).
- For \( t = 1, \ldots, T \), do:
  - Give \( D_t \)-weighted examples to Weak Learner; get back \( h_t: \mathcal{X} \to \{-1, +1\} \).
  - Compute weight on \( h_t \) and update weights on examples:
    \[
    s_t := \sum_{i=1}^n D_t(i) \cdot y_i h_t(x_i) \\
    \alpha_t := \frac{1}{2} \ln \frac{1 + s_t}{1 - s_t} \\
    D_{t+1}(i) := \frac{D_t(i) \cdot \exp(-\alpha_t \cdot y_i h_t(x_i))}{Z_t} 
    \]
    for each \( i = 1, \ldots, n \)
    
    where
    \[
    Z_t := \sum_{i=1}^n D_t(i) \cdot \exp(-\alpha_t \cdot y_i h_t(x_i))
    \]
    is the normalizer that makes \( D_{t+1} \) a probability distribution.
- Final hypothesis is \( \hat{h} \) defined by \( \hat{h}(x) := \text{sign} \left( \sum_{t=1}^T \alpha_t \cdot h_t(x) \right) \) for \( x \in \mathcal{X} \).

Training error rate bound
Let \( \hat{\ell} \) be the function defined by
\[
\hat{\ell}(x) := \sum_{t=1}^T \alpha_t \cdot h_t(x) \quad \text{for } x \in \mathcal{X}
\]
so \( \hat{h}(x) = \text{sign}(\hat{\ell}(x)) \). The training error rate of \( \hat{h} \) can be bounded above by the average exponential loss of \( \hat{\ell} \):
\[
\frac{1}{n} \sum_{i=1}^n \mathbf{1}\{\hat{h}(x_i) \neq y_i\} \leq \frac{1}{n} \sum_{i=1}^n \exp(-y_i \hat{\ell}(x_i)).
\]
This holds because
\[
\hat{h}(x_i) \neq y_i \iff -y_i \hat{\ell}(x_i) \geq 0 \iff \exp(-y_i \hat{\ell}(x_i)) \geq 1.
\]
Furthermore, the average exponential loss of $\hat{\ell}$ equals the product of the normalizers from all rounds:

$$\frac{1}{n} \sum_{i=1}^{n} \exp(-y_i \hat{\ell}(x_i)) = \sum_{i=1}^{n} D_1(i) \cdot \exp \left( - \sum_{t=1}^{T} \alpha_t \cdot y_i h_t(x_i) \right)$$

$$= Z_1 \sum_{i=1}^{n} D_2(i) \cdot \exp \left( - \sum_{t=2}^{T} \alpha_t \cdot y_i h_t(x_i) \right)$$

$$= Z_1 Z_2 \sum_{i=1}^{n} D_3(i) \cdot \exp \left( - \sum_{t=3}^{T} \alpha_t \cdot y_i h_t(x_i) \right)$$

$$= \ldots$$

$$= \prod_{t=1}^{T} Z_t.$$ 

Since each $y_i h_t(x_i) \in \{-1, +1\}$, the normalizer $Z_t$ can be written as

$$Z_t = \sum_{i=1}^{n} D_t(i) \cdot \exp(-\alpha_t \cdot y_i h_t(x_i))$$

$$= \sum_{i=1}^{n} D_t(i) \cdot \left( \frac{1 + y_i h_t(x_i)}{2} \exp(-\alpha_t) + \frac{1 - y_i h_t(x_i)}{2} \exp(\alpha_t) \right)$$

$$= \sum_{i=1}^{n} D_t(i) \cdot \left( \frac{1 + y_i h_t(x_i)}{2} \sqrt{\frac{1 - s_t}{1 + s_t}} + \frac{1 - y_i h_t(x_i)}{2} \sqrt{\frac{1 + s_t}{1 - s_t}} \right)$$

$$= \sqrt{(1 + s_t)(1 - s_t)}$$

$$= \sqrt{1 - s_t^2}.$$ 

So, we conclude the following bound on the training error rate of $\hat{h}$:

$$\frac{1}{n} \sum_{i=1}^{n} 1\{\hat{h}(x_i) \neq y_i\} \leq \frac{1}{n} \sum_{i=1}^{n} \exp(-y_i \hat{\ell}(x_i)) = \prod_{t=1}^{T} Z_t = \prod_{t=1}^{T} \sqrt{1 - s_t^2} \leq \exp \left( -\frac{1}{2} \sum_{t=1}^{T} s_t^2 \right)$$

where the last step uses the fact that $1 + x \leq e^x$ for any real number $x$.

(The bound is usually written in terms of $\gamma_t := s_t/2$, i.e., as $\exp(-2 \sum_{t=1}^{T} \gamma_t^2)$.)

**Margins on training examples**

Let $\hat{g}$ be the function defined by

$$\hat{g}(x) := \frac{\sum_{t=1}^{T} \alpha_t \cdot h_t(x)}{\sum_{t=1}^{T} |\alpha_t|} \quad \text{for} \quad x \in \mathcal{X}$$

so $y_i \hat{g}(x_i)$ is the margin achieved on example $(x_i, y_i)$. We may assume without loss of generality that $\alpha_t \geq 0$ for each $t = 1, \ldots, T$ (by replacing $h_t$ with $-h_t$ as needed.) Fix a value $\theta \in (0, 1)$, and consider the fraction of training examples on which $\hat{g}$ achieves a margin at most $\theta$:

$$\frac{1}{n} \sum_{i=1}^{n} 1\{y_i \hat{g}(x_i) \leq \theta\}.$$ 

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This quantity can be bounded above using the arguments from before:

\[
\frac{1}{n} \sum_{i=1}^{n} 1\{y_i \hat{g}(x_i) \leq \theta\} = \frac{1}{n} \sum_{i=1}^{n} 1\{y_i \hat{\ell}(x_i) \leq \theta \sum_{t=1}^{T} \alpha_t\}
\]

\[
\leq \exp\left(\theta \sum_{t=1}^{T} \alpha_t\right) \cdot \frac{1}{n} \sum_{i=1}^{n} \exp(-y_i \hat{\ell}(x_i))
\]

\[
= \exp\left(\theta \sum_{t=1}^{T} \alpha_t\right) \cdot \prod_{t=1}^{T} \sqrt{1 - s_t^2}
\]

\[
= \prod_{t=1}^{T} \sqrt{(1 + s_t)^{1+\theta}(1 - s_t)^{1-\theta}}.
\]

Suppose that for some $\gamma > 0$, $s_t \geq 2\gamma$ for all $t = 1, \ldots, T$. If $\theta < \gamma$, then using calculus, it can be shown that each term in the product is less than 1:

\[
\sqrt{(1 + s_t)^{1+\theta}(1 - s_t)^{1-\theta}} < 1.
\]

Hence, the bound decreases to zero exponentially fast with $T$. 