Lagrange interpolation

COMS 3251 Fall 2022 (Daniel Hsu)

1 Lagrange polynomials

Recall that $\mathsf{P}_n(\mathbb{R})$, the set of univariate polynomials with real coefficients and degree at most n, is a vector space of dimension $n+1$. A commonly used basis for $\mathsf{P}_n(\mathbb{R})$ is $\{1, t, t^2, \dots, t^n\}$; this can be regarded as an analogue of the "standard basis" for $\mathbb{R}^{n+1}$. However, other bases for $\mathsf{P}_n(\mathbb{R})$ may be more convenient in certain circumstances.

The *Lagrange polynomials* form a "data-driven" basis for $\mathsf{P}_n(\mathbb{R})$. Given a dataset of $n+1$ distinct real numbers $x_0, x_1, \dots, x_n$, the Lagrange polynomials $\mathcal{L} = \{\ell_0(t), \ell_1(t), \dots, \ell_n(t)\}$ are defined by[1]

$$\ell_i(t) \;=\; \frac{(t-x_0)\cdots(t-x_{i-1})(t-x_{i+1})\cdots(t-x_n)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)} \;=\; \prod_{j\in\{0,\dots,n\}\setminus\{i\}} \frac{t-x_j}{x_i-x_j}.$$

Notice that the numerator of $\ell_i(t)$ has exactly n factors: $(t-x_i)$ does not appear. Also notice that the denominator of $\ell_i(t)$ has exactly n factors and is non-zero: $(x_i - x_j) \neq 0$ for all $j \neq i$, and $(x_i - x_i)$ does not appear. So each $\ell_i(t)$ is, indeed, a univariate polynomial with real coefficients, and expanding the product of factors in the numerator reveals that the degree of $\ell_i(t)$ is n.

2 Properties of Lagrange polynomials

For any dataset of n distinct real numbers $x_0, x_1, \dots, x_n$, the Lagrange polynomials have the following remarkable data-dependent property:

$$\ell_i(x_j) \;=\; \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{if } j \neq i. \end{cases} \tag{1}$$

To see this, consider any $\ell_i(t)$. When we evaluate $\ell_i(t)$ at $t = x_i$, the numerator and denominator in the definition of $\ell_i(t)$ become the same, so the value

[1] It would have been better to use notation that reminds the reader that each polynomial $\ell_i(t)$ depends on the dataset $x_0, x_1, \dots, x_n$.

is 1. But when we evaluate $\ell_i(t)$ at $t = x_j$ for $j \neq i$, the numerator includes $(x_j - x_j)$ as a term, so the product is zero.

Using this property, it is easy to see why the Lagrange polynomials form a basis for $\mathsf{P}_n(\mathbb{R})$.

Proposition 1. *$\mathcal{L}$ is a linearly independent set of $n+1$ polynomials from $\mathsf{P}_n(\mathbb{R})$.*

Proof. We show that any linear combination of $\ell_0(t), \ell_1(t), \ldots, \ell_n(t)$ that results in the zero polynomial (denote by $\text{zero}(t)$) must be the all-zeros linear combination.

Suppose there are scalars $c_0, c_1, \ldots, c_n$ such that

$$c_0\,\ell_0(t) + c_1\,\ell_1(t) + \cdots + c_n\,\ell_n(t) \;=\; \text{zero}(t). \tag{2}$$

We evaluate the left-hand side of (2) at various real numbers; of course, evaluating the right-hand side of (2), the zero polynomial, at any real number gives a result of 0. When we evaluate the left-hand side of (2) at $t = x_i$, we obtain

$$c_0\,\ell_0(x_i) + c_1\,\ell_1(x_i) + \cdots + c_n\,\ell_n(x_i) \;=\; c_i$$

by (1). Since the right-hand side of (2) is 0 at $t = x_i$, it follows that $c_i = 0$. Since this holds for any $i \in \{0, 1, \ldots, n\}$, it follows that $c_0 = c_1 = \cdots = c_n = 0$. So we have proved that $\ell_0(t), \ell_1(t), \ldots, \ell_n(t)$ are linearly independent. □

Since $\mathcal{L}$ is a linearly independent set of $n+1$ polynomials from $\mathsf{P}_n(\mathbb{R})$, it follows by the Basis Sufficiency Theorem that $\mathcal{L}$ is a basis for $\mathsf{P}_n(\mathbb{R})$.

3 Coordinate representations

Since $\mathcal{L}$ is a basis for $\mathsf{P}_n(\mathbb{R})$, we can write any polynomial $p(t) \in \mathsf{P}_n(\mathbb{R})$ as a linear combination of the Lagrange polynomials corresponding to a given dataset $x_0, x_1, \ldots, x_n$:

$$p(t) \;=\; a_0\,\ell_0(t) + a_1\,\ell_1(t) + \cdots + a_n\,\ell_n(t)$$

for some choice of scalars $a_0, a_1, \ldots, a_n$. In fact, it is easy to see what these scalars are, again by using (1):

$$p(x_i) \;=\; a_0\,\ell_0(x_i) + a_1\,\ell_1(x_i) + \cdots + a_n\,\ell_n(x_i) \;=\; a_i.$$

So the coordinate representation of $p(t)$ with respect to $\mathcal{L}$ is

$$[p(t)]_{\mathcal{L}} \;=\; (p(x_0), p(x_1), \dots, p(x_n)).$$

By the Unique Representations Theorem, for any $(y_0, y_1, \dots, y_n) \in \mathbb{R}^{n+1}$, there is a unique polynomial $p(t) \in \mathsf{P}_n(\mathbb{R})$ such that

$$p(x_i) \;=\; y_i \quad \text{for all } i \in \{0, 1, \dots, n\};$$

the polynomial $p(t)$ is simply

$$p(t) \;=\; y_0\,\ell_0(t) + y_1\,\ell_1(t) + \cdots + y_n\,\ell_n(t).$$

This way of mapping from $n+1$ pairs of real numbers $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$ (with the only restriction being that $x_0, x_1, \dots, x_n$ must be distinct) to a polynomial is called *Lagrange interpolation*.

One final consequence of Lagrange polynomials: If $p(t) \in \mathsf{P}_n(\mathbb{R})$ evaluates to 0 at $n+1$ distinct real numbers, then $p(t)$ must be the zero polynomial.