Classification using generative models

COMS 4771 Fall 2023

Classification problems

Problem: Create a program that, given an element from the input space $\mathcal{X}$, returns the element's corresponding label from the output space $\mathcal{Y}$

Classification problem: $\mathcal{Y}$ is discrete (and typically finite) set

Examples:

- Spam filtering

$$\mathcal{X} = \{\text{all emails}\}, \quad \mathcal{Y} = \{\text{ham}, \text{spam}\}$$

- Intrusion detection

$$\mathcal{X} = \{\text{all network traffic logs}\}, \quad \mathcal{Y} = \{\text{benign}, \text{malicious}\}$$

- News analysis

$$\mathcal{X} = \{\text{all news articles}\}, \quad \mathcal{Y} = \{\text{politics}, \text{sports}, \text{business}, \ldots\}$$

Model data as random variables

- Feature vector: random vector $X = (X_1, X_2, \ldots, X_d)$
- Label: a discrete random variable Y
- X and Y may be dependent!

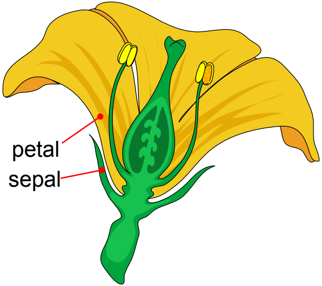
Typical goal: create a classifier $f \colon \mathcal{X} \to \mathcal{Y}$ with low error rate

$$\operatorname{err}[f] = \Pr(f(X) \neq Y)$$

Iris dataset

Iris dataset (Fisher, 1936)

- 3 classes of iris plants $\mathcal{Y} = \{\text{Setosa}, \text{Versicolor}, \text{Virginica}\}$
- Take some measurements of each iris plant



- Training data: 40 examples from each class; test data: 10 examples per class
- Problem: Create a program that, given the measurements of an iris plant, returns the class that the plant belongs to

Generative models for classification

Generative model (for classification): a family of probability distributions for (X, Y), each with the following form:

- Specify marginal distribution p_Y of Y (class prior)
- For each $k \in \mathcal{Y}$, specify conditional distribution of X given $Y = k$ (class conditional distributions)

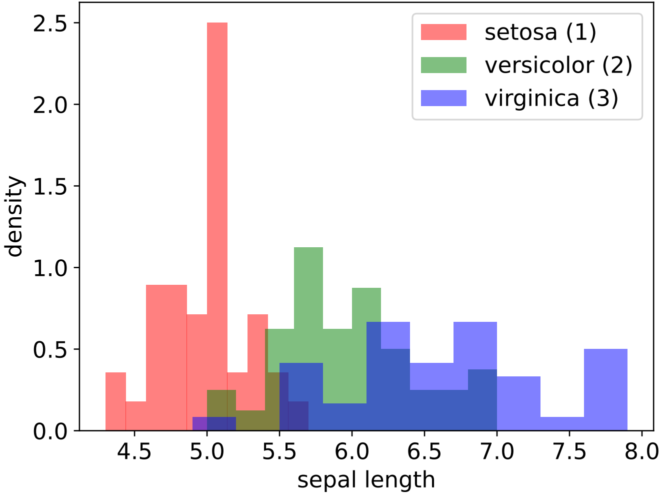
Example: Normal generative model

How to create classifier based on a distribution from the generative model?

- You have: $\hat{p}_Y$ and $\hat{p}_{X|Y=k}$ for each $k \in \mathcal{Y}$
- You want: $\hat{f}\colon \mathcal{X} \to \mathcal{Y}$

Generative model for iris dataset

Normal generative model for iris dataset using x = sepal length



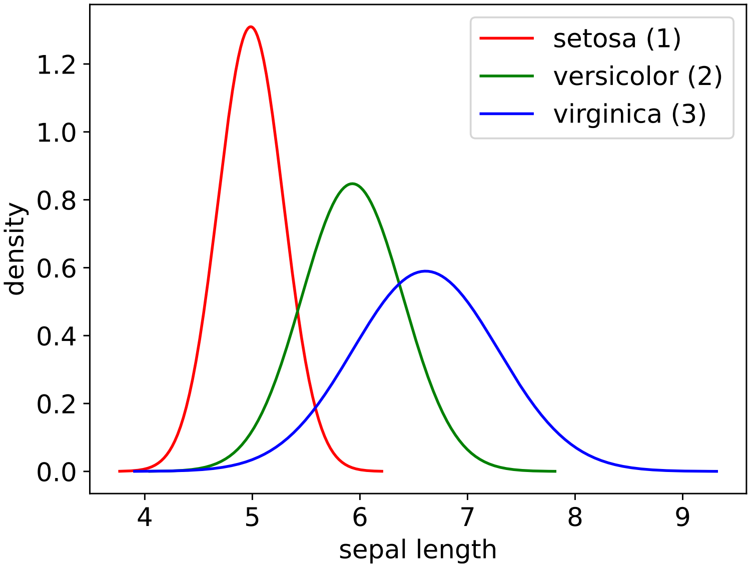
Maximum likelihood estimation (MLE) of π_k, μ_k, σ_k^2 for each $k \in \mathcal{Y}$:

$$\hat{\pi}_k = \frac{\text{\# training examples with label } k}{\text{\# training examples}}$$

$$\hat{\mu}_k = \text{average value of } x \text{ among examples with label } k$$

$$\hat{\sigma}_k^2 = \text{average value of } (x - \hat{\mu}_k)^2 \text{ among examples with label } k$$

k	setosa (1)	versicolor (2)	virginica (3)
$\hat{\pi}_k$	$1/3$	$1/3$	$1/3$
$\hat{\mu}_k$	4.99	5.93	6.61
$\hat{\sigma}_k$	0.31	0.47	0.68

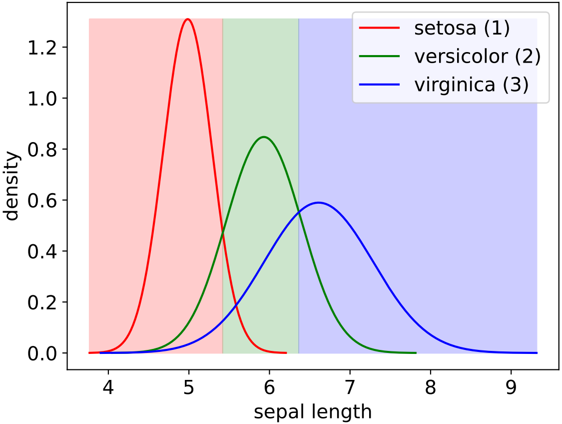

setosa (1)
versicolor (2)
virginica (3)
density
1.2
1.0
0.8
0.6
0.4
0.2
0.0
4
5
6
7
8
9
sepal length

```python
def learn(train_x, train_y, num_classes=3):
  return [(np.mean(train_y == k), np.mean(train_x[train_y == k]),
  ↪  np.var(train_x[train_y == k])) for k in range(num_classes)]

def predict(params, test_x):
  log_posterior = np.array([np.log(prior) - np.log(sigma2) / 2 -
  ↪  (test_x - mu) ** 2 / (2*sigma2) for prior, mu, sigma2 in
  ↪  params])
  return np.argmax(log_posterior, axis=0)
```

Resulting classifier:

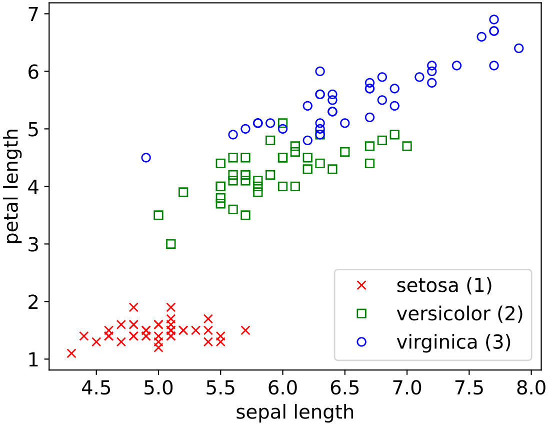
$$\hat{f}(x) = \arg\max_{k \in \mathcal{Y}} \hat{\pi}_k \cdot \frac{1}{\sqrt{2\pi\hat{\sigma}_k^2}} \exp\left(-\frac{(x-\hat{\mu}_k)^2}{2\hat{\sigma}_k^2}\right)$$



Training error rate of $\hat{f}$: 24%
Test error rate of $\hat{f}$: 40%

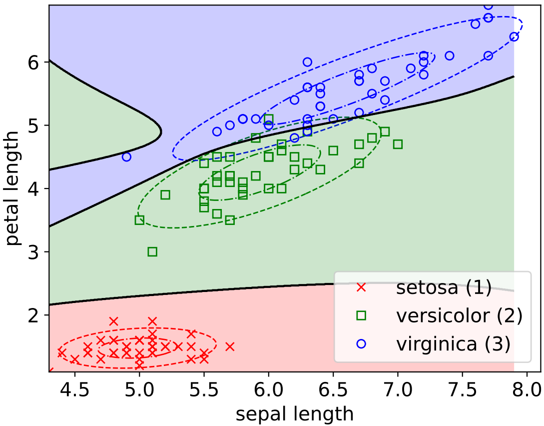
Bivariate normal distributions

Now use two features: $x = (x_1, x_2) = (\text{sepal length}, \text{petal length})$



Need generative model with class conditional distributions suitable for two-dimensional feature vectors

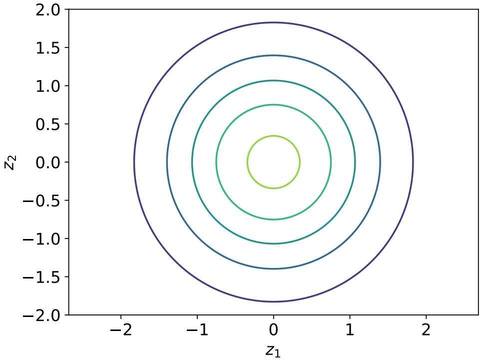
Bivariate normal distribution: 5 parameters (up from 2)



Resulting classifier has test error rate 10% (down from 40%)

If Z_1 and Z_2 are independent random variables and each is a standard normal random variable, then distribution of $Z = (Z_1, Z_2)$ is standard bivariate normal

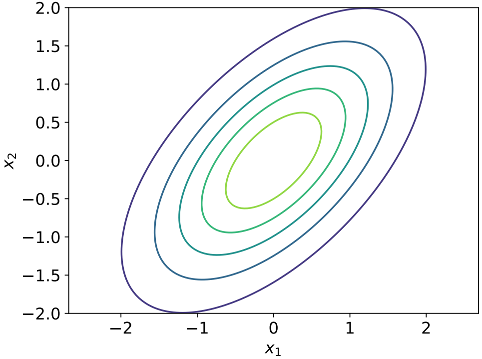
$$p_{(Z_1,Z_2)}(z_1, z_2) = \frac{1}{2\pi} \exp\left(-\frac{z_1^2 + z_2^2}{2}\right) = \phi(z_1)\phi(z_2)$$



What is distribution of $X = aZ_1 + bZ_2$?

What is distribution of (X_1, X_2), with $X_1 = aZ_1 + bZ_2$ and $X_2 = cZ_1 + dZ_2$?

Density function for (X_1, X_2) where $X_1 = Z_1 + \frac{1}{3}Z_2$ and $X_2 = \frac{1}{3}Z_1 + Z_2$:



General bivariate normal distribution:

$$p_{(X_1,X_2)}(x_1, x_2) = \frac{1}{2\pi\sqrt{\det(\Sigma)}} \exp\left(-\frac{1}{2} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix}^\mathsf{T} \Sigma^{-1} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix} \right)$$

Fitting bivariate normal distribution to data using MLE:

$$\hat{\mu}_1 = \text{average value of } x_1 \text{ in dataset}$$
$$\hat{\mu}_2 = \text{average value of } x_2 \text{ in dataset}$$
$$\hat{\Sigma}_{1,1} = \text{average value of } (x_1 - \hat{\mu}_1)^2 \text{ in dataset}$$
$$\hat{\Sigma}_{1,2} = \hat{\Sigma}_{2,1} = \text{average value of } (x_1 - \hat{\mu}_1)(x_2 - \hat{\mu}_2) \text{ in dataset}$$
$$\hat{\Sigma}_{2,2} = \text{average value of } (x_2 - \hat{\mu}_2)^2 \text{ in dataset}$$

(In context of generative models, do this for each class)

Multivariate normal distributions

General multivariate normal distribution in d-dimensions:

$$p_X(x) = \frac{1}{\sqrt{(2\pi)^d \det(\Sigma)}} \exp\left(-\frac{1}{2}(x-\mu)^\intercal \Sigma^{-1}(x-\mu) \right)$$

MLE:

$$\hat{\mu}_i = \text{average value of } x_i \text{ in dataset}$$

$$\hat{\Sigma}_{i,j} = \text{average value of } (x_i - \hat{\mu}_i)(x_j - \hat{\mu}_j) \text{ in dataset}$$