COMS 4773 Fall 2026 Problem Set 0

Problem 1. Let X be a non-negative integer-valued random variable.

(a) Prove that $\Pr(X \neq 0) \leq \mathrm{E}(X)$.

(b) Prove that $\Pr(X = 0) \leq \mathrm{var}(X) / \mathrm{E}(X^2)$.

Problem 2. We say (X, d) is a *metric space* if X is a set of points and $\mathsf{d} \colon X \times X \to \mathbb{R}$ is a function that satisfies the following:

1. $\mathsf{d}(x, y) \geq 0$ for all $x, y \in X$, with equality if and only if $x = y$ (positivity);
2. $\mathsf{d}(x, y) = \mathsf{d}(y, x)$ for all $x, y \in X$ (symmetry);
3. $\mathsf{d}(x, z) \leq \mathsf{d}(x, y) + \mathsf{d}(y, z)$ for all $x, y, z \in X$ (triangle inequality).

For any $\epsilon > 0$, we say $C \subseteq X$ is an *ϵ-cover* of (X, d) if, for each $x \in X$, there exists $\tilde{x} \in C$ such that $\mathsf{d}(x, \tilde{x}) \leq \epsilon$. Also, for any $\epsilon > 0$, we say $P \subseteq X$ is an *ϵ-packing* of (X, d) if $\mathsf{d}(x, y) > \epsilon$ for all distinct $x, y \in P$.

(a) Suppose (X, d) is a metric space, and $P \subseteq X$ satisfies the following:

   - P is an ϵ-packing of (X, d), and
   - for all $x \in X \setminus P$, $P \cup \{x\}$ is not an ϵ-packing of (X, d).

   Prove that P is an ϵ-cover of (X, d).

(b) Suppose (X, d) is a metric space, and for some $\epsilon > 0$, C is an ϵ-cover of (X, d), and P is a 2ϵ-packing of (X, d). Prove that $|P| \leq |C|$.

Problem 3. We say $\|\cdot\| \colon \mathbb{R}^n \to \mathbb{R}$ is a *norm* if it satisfies the following:

1. $\|x\| \geq 0$ for all $x \in \mathbb{R}^n$, with equality if and only if $x = 0$ (positivity);
2. $\|tx\| = |t|\|x\|$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$ (homogeneity);
3. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathbb{R}^n$ (triangle inequality).

For $p \geq 1$, the l^p norm for vectors in $\mathbb{R}^n$ is defined by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \quad \text{for all } x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

(In this class, when the subscript p in $\|\cdot\|_p$ is omitted, the Euclidean norm case $p = 2$ is typically assumed.) For $p = \infty$, the l^∞ norm is defined by

$$\|x\|_\infty = \max\{|x_i| \mid i \in \{1, \ldots, n\}\} \quad \text{for all } x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

(a) Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$, and define $\mathsf{d}(x, y) := \|x - y\|$ for all $x, y \in \mathbb{R}^n$. Prove that $(\mathbb{R}^n, \mathsf{d})$ is a metric space.

(b) Prove that for any vectors $x, y \in \mathbb{R}^n$,

$$\langle x, y\rangle \leq \|x\|_1 \|y\|_\infty.$$

(Here, $\langle \cdot, \cdot \rangle$ is the standard inner product between n-vectors: for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, we have $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$.)

(c) Prove that $\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2$ for all $x \in \mathbb{R}^n$.

Problem 4. The *convex hull* of the vectors $v_1, \ldots, v_m \in \mathbb{R}^n$ is the set

$$\left\{ \sum_{i=1}^{m} \alpha_i v_i \;\middle|\; \alpha_1, \ldots, \alpha_m \geq 0,\ \sum_{i=1}^{m} \alpha_i = 1 \right\}.$$

Prove that for any vectors $v_1, \ldots, v_m \in \mathbb{R}^n$, each with $\|v_i\|_2 \leq 1$, any positive integer k, and any vector u in the convex hull of $v_1, \ldots, v_m$, there exists non-negative integers $k_1, \ldots, k_m$ with $\sum_{i=1}^m k_i = k$ such that

$$\left\| u - \frac{1}{k} \sum_{i=1}^{m} k_i v_i \right\|_2^2 \leq \frac{1}{k}.$$