The Interval Packing Process of Linear Networks

E. G. Coffman, Jr.[1], Ph. Robert[2] and A. L. Stolyar[1]
1. Bell Labs, Lucent Technologies, 700 Mountain Ave., Murray Hill, NJ 07974
2. INRIA, Domaine de Voluceau, B.P. 105, 78153 Le Chesnay Cedex, France.

Introduction. Start with the following elegant interval packing (IP) problem. Let random subintervals arrive at a service facility in a rate-λ Poisson stream; a random subinterval is simply the interval between two independent uniform random draws from $[0,1]$. Place an arrival at time t immediately into service if it is disjoint from each of the subintervals, if any, being served at time t; otherwise, place it into a queue. When a subinterval completes service, at time t say, scan the queue in arrival order, placing into service each subinterval encountered that is disjoint from all subintervals already accepted for service at time t. *Determine the stability condition under the assumption that service times are independent unit-mean exponentials.*

A little reflection shows that, since the midpoint $1/2$ is the most "congested" point of $[0,1]$ and since the probability that a random subinterval covers the midpoint is $1/2$, then $\lambda < 2$ is a necessary condition for stability. Thus, our problem reduces to determining whether $\lambda < 2$ is sufficient.

Although we have been unable to answer this question precisely as formulated, we *have* been able to answer discretized versions, which is sufficient for practical purposes. In the discretized version with parameter m, a random subinterval is a discrete interval $[i,j]$, $0 \leq i < j \leq m$, where i and j are the smaller and larger of two independent uniform random draws from the set $\{0,\ldots,m\}$. In this case, the above argument shows that

$$\lambda < \frac{2\lfloor \frac{m}{2} \rfloor + 1}{\lfloor \frac{m}{2} \rfloor + 1} \tag{1}$$

is a necessary condition for stability, which we note has the limit $\lambda < 2$ as expected when $m \to \infty$. It was shown in [1] that (1) is indeed sufficient for stability when $m = 2$, the least value of m for which the problem is nontrivial. We have extended this result to

Theorem 1 *If (1) holds for $m = 3$, then the IP process on $[0,3]$ is ergodic.*

What is more, we believe that our proof techniques can be extended to larger m. Before getting into the foundations of the proof, we say a word about applications.

Applications. In linear interconnection networks, IP models the stochastic arrival and departure of connections or calls. An interval $[i,j]$ is a communication path between nodes i and j of a network consisting of a chain of m links. Loss networks of this general type have received considerable study (see, e.g., the review by Kelly [4]), while the linear networks with queueing that are considered here have been studied more recently in [1].

Proof highlights. The fluid-limit approach originated by Rybko and Stolyar [6] has been very successful in dealing with stability problems of the sort presented here. In what follows, we describe this approach in somewhat broad, informal terms as it applies to the proof of Theorem 1. Many technical details are omitted which the reader can find in [2] and [3].

Recall that the fluid limit of a process $(X(t), t \geq 0)$, assuming it exists, is the limit (in an appropriate sense) of the n-th scaled process $(\frac{1}{n}X(nt), t \geq 0)$ as $n \to +\infty$ when the norm of the initial state $X(0)$ is equal to $\alpha n + o(n)$ for some fixed $\alpha \geq 0$. For example, let $\lambda_{i,j}$ denote the arrival rate of intervals $[i,j]$. Then the fluid limit $f_{i,j}(t)$ of our Poisson arrival of intervals $[i,j]$ is a straight line of slope $\lambda_{i,j}$. We assume that the n-th scaled IP process is such that the initial queue length is n, and hence that the corresponding norm $q(t)$ of a fluid limit, if one exists, has the initial value $q(0) = 1$. To prove ergodicity the objective is to show that $q(t)$ drifts from 1 to 0 in finite time. For this, it is sufficient to show that there exists a $T > 0$ such that $q(T) < 1 - \varepsilon$ for some fixed $\varepsilon > 0$. In this case, a classical criterion (see Malyshev and Menshikov [5]) would show that the IP process is ergodic. (See [2, 3] for versions of this result more useful for our purposes.)

If $\hat{f}_{i,j}(t)$ denotes the fluid limit of the departure process of intervals $[i, j]$, then the family of fluid limits $(f_{i,j}(t), \hat{f}_{i,j}(t),\ t \geq 0)$ comprises the fluid limit of the Markov IP process, and defines (by projection) the fluid limit $q(t)$ of the queue length process. To prove the drift criterion above, define $\tau_{i,j}(t)$ as the fluid limit of the least time such that the intervals $[i, j]$ currently in the system at time t arrived during $[\tau_{i,j}(t), t]$. More precisely, $\tau_{i,j}(t) := f_{i,j}^{-1}(\hat{f}_{i,j}(t))$ with $f_{i,j}(t)$ suitably extended to $(-\infty, 0)$ so that $\tau_{i,j}(t)$ is well defined for all $t \geq 0$ (see [2] for details). The following two lemmas are the basis of the proof of Theorem 1. The first is an easy consequence of the simple fact that, if $[i, j] \subseteq [k, l]$, then $[i, j]$ is easier to pack (fits into more configurations) than $[k, l]$.

Lemma 1 *Consider the fluid limit of the IP process for any given $m > 0$. There exists a $T_1 > 0$ such that $[i, j] \subseteq [k, l]$ implies $\tau_{i,j}(t) \geq \tau_{k,l}(t)$ for all $t \geq T_1$.*

Let $q_{\overline{i,j}}(t)$ be the fluid limit of the total number of intervals in queue that cover the interval $[i, j]$. Similarly, let $\lambda_{\overline{i,j}}$ and $\mu_{\overline{i,j}}(t)$ be the total arrival and departure rates of the intervals covering $[i, j]$ in the fluid limit. Note that the μ's vary with time in general.

Lemma 2 *For $m = 3$ there exists an $\epsilon > 0$ such that at any regular point $t > T_1$, the inequality $q_{\overline{1,2}}(t) > 0$ implies $\mu_{\overline{1,2}}(t) > \lambda_{\overline{1,2}} + \epsilon$.*

This result takes more effort to prove. To motivate it, consider the original system and observe that, just as soon as an interval $[1, 2]$ begins service, intervals $[0, 1], [1, 2]$, and $[2, 3]$ will be served together until there are no more of one of the three types left to serve. Note also that, in addition to the above configuration, the outer intervals $[0, 1]$ and $[2, 3]$ can be served with intervals $[1, 3]$ and $[0, 2]$ respectively. Thus, one expects that the system will spend relatively little of its time in states having one or both of the outer unit intervals only (i.e., in states where $[1, 2]$ is not covered), and that therefore, in the fluid limit, the effect of these states will not survive and Lemma 2 will hold.

References

[1] E. G. Coffman, Jr., A. Feldmann, N. Kahale, B. Poonen. Computing Call Admission Capacities in Linear Networks. *Prob. Eng. Inf. Sci.*, to appear.

[2] E. G. Coffman, Jr. and A. L. Stolyar. Bandwidth Packing. *Algorithmica*, to appear.

[3] Jean-Francois Dantzer, Mostafa Haddani, and Philippe Robert. On the stability of a bandwidth packing algorithm. Technical Report, INRIA, Domaine de Voluceau, B.P. 105, 78153 Le Chesnay Cedex, France.

[4] F. P. Kelly. Loss networks. *Ann. Appl. Prob.*, **1**(1991), pages 319–378.

[5] V.A. Malyshev and M.V. Menshikov. Ergodicity, Continuity, and Analyticity of Countable Markov Chains. *Transactions of Moscow Mathematical Society*, Vol. 39, (1979), pp. 3-48.

[6] A.N. Rybko and A.L. Stolyar. Ergodicity of Stochastic Processes Describing the Operation of Open Queueing Networks *Problems of Information Transmission*, Vol. 28, (1992), pp. 199-220.