MEASUREMENTS OF INTERDEPENDENCE

RON BLEI

Our theme here is that interdependence can be calibrated by indices based, separately, on combinatorial measurements, p-variations, and tail-probability estimates. These notions – that each of the aforementioned is linked to interdependence – had naturally originated in a context of harmonic analysis ([1], [2]), and appeared later in stochastic settings (e.g., [12], [4], [6], [5]). Eventually this (and more) was detailed in [7] – too large a book (alas!), which at this point almost surely could use a revision... Meanwhile, I intend here to survey and explain these ideas, and (hopefully) also shed some new light on them. Because my audience appears fairly heterogenous, I will stick to genesis, and will discuss interdependence essentially in the same basic context where I first encountered it. No formal proofs will be given. I will speak heuristically, but will try to be precise.

1. Functional interdependence – basic notions

Let $d \geqslant 2$ be an integer. Suppose F and $E_j,\ j \in [d]$, are sets, and

$$f_j : F \to E_j,\ j \in [d],$$

are given functions, which we may as well assume to be *onto*. (As usual, $[d]$ denotes $\{1, \dots, d\}$.) Nothing is known, or assumed, about structures in F and E_j.

Question 1.1. *How interdependent are $f_1, \dots, f_d$?*

Of course we need to say what we mean by *interdependence* and how we measure it. We begin with the observation that the *interdependence* of $f_1, \dots, f_d$ is expressed, somehow, by the inclusions

$$\big\{\big(f_i(t)\big)_{i\in T} : t \in \bigcap_{i\in S}\{f_i = y_i\}\big\} \subset \bigtimes_{j\in T} E_j, \tag{1.1}$$

where $S \subsetneq [d]$, $(y_i : i \in S) \in \bigtimes_{i\in S} E_i$, and $T \subset [d]\backslash S$. (As usual, $\{f_i = y_i\}$ is the abbreviation for $\{t \in F : f_i(t) = y_i\}$.) To wit, *interdependence* is detected, somehow, by the "extent" that the left side of (1.1) differs from the full Cartesian product $\bigtimes_{j\in T} E_j$ on the right side.

To illustrate this, and also suggest how to proceed, we consider two extremal instances. In the first, (1.1) is an equality for $S = \{j\}$ for every $j \in [d]$ and all $y \in E_j$. In this case, $f_1, \dots, f_d$ are viewed as *independent* functions: for, here the knowledge that $f_j = y$, $y \in E_j$, implies *no* information about the values of f_i for $i \neq j$. In a second example, opposite to the first, there

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exists $j_0 \in [d]$ such that the left side of (1.1) is a singleton for every $y \in E_{j_0}$: here, evaluations of f_{j_0} uniquely determine the values of f_i for every $i \in [d]$.

Definition 1.1. *For a positive integer $d \geqslant 2$, let $f_1, \ldots, f_d$ be functions from a set F onto sets $E_1, \ldots, E_d$, respectively.*

I. $f_1, \ldots, f_d$ are functionally independent if for every

$$(y_1, \ldots, y_d) \in E_1 \times \cdots \times E_d,$$

there exists $t \in F$ such that

$$f_j(t) = y_j, \quad j \in [d].$$

II. $f_1, \ldots, f_d$ are functionally dependent if there exists $j_0 \in [d]$ such that for all $y \in E_{j_0}$,

$$\#\big\{\big(f_i(t)\big)_{i \neq j_0} : t \in \{f_{j_0} = y\}\big\} = 1.$$

(# denotes cardinality.)

Given that $f_1, \ldots, f_d$ could be neither functionally independent nor functionally dependent, our task will be to calibrate, somehow, the "gap" between these two extremes.

To motivate what we do, consider the function

$$\mathbf{f}_{[d]} = (f_1, \ldots f_d) : \ F \to E_1 \times \cdots \times E_d, \tag{1.2}$$

and its image,

$$\mathbf{f}_{[d]}(F) = \big\{\big(f_1(t), \ldots, f_d(t)\big) : t \in F\big\}. \tag{1.3}$$

Note that *functional independence* means simply that the image of $\mathbf{f}_{[d]}$ is the full Cartesian product,

$$\mathbf{f}_{[d]}(F) = E_1 \times \cdots \times E_d,$$

which is $d-$*dimensional*. At the other end, *functional dependence* means that for some $j_0 \in [d]$ there exist functions

$$\theta_i : E_{j_0} \to E_i, \quad i \in [d],$$

such that

$$f_i = \theta_i \circ f_{j_0}.$$

Assuming $j_0 = 1$, let

$$\Theta = (\theta_2, \ldots, \theta_d) : E_1 \to E_2 \times \cdots \times E_d, \tag{1.4}$$

and note that the image

$$\mathbf{f}_{[d]}(F) = \operatorname{graph} \Theta = \big\{\big(y, \Theta(y)\big) : y \in E_1\big\}$$

in this case is a *one-dimensional* "curve" (parameterized by E_1) in $E_1 \times \cdots \times E_d$. (Here we still think of "dimension" intuitively, as the number of "degrees of freedom.")

To view instances of *interdependence* between the two respective extremes, take $f_1, \ldots, f_d$ with functionally independent $f_1, \ldots, f_k$ $(1 < k < d)$, such that for all $(y_1, \ldots, y_k) \in E_1 \times \cdots \times E_k$,

$$\#\{\big(f_i(t) : i > k\big) : t \in \bigcap_{i=1}^{k} \{f_i = y_i\}\} = 1. \tag{1.5}$$

The "count" in (1.5) means that there are functions

$$\theta_i : E_1 \times \cdots \times E_k \to E_i, \quad i \in [d],$$

such that

$$f_i = \theta_i \circ \mathbf{f}_{[k]}, \tag{1.6}$$

where

$$\mathbf{f}_{[k]} = (f_1, \ldots, f_k) : F \to E_1 \times \cdots \times E_k.$$

(Cf. (1.2).) (For $i \in [k]$, θ_i is the canonical projection π_i from $E_1 \times \cdots \times E_k$ onto E_i:

$$\pi_i(\mathbf{y}) = y_i, \quad \mathbf{y} = (y_j)_{j \in [k]}.) \tag{1.7}$$

Let (cf. (1.4))

$$\Theta_k = (\theta_{k+1}, \ldots, \theta_d) : E_1 \times \cdots \times E_k \to E_{k+1} \times \cdots \times E_d, \tag{1.8}$$

and then note (because $f_1, \ldots, f_k$ are functionally independent) that the image

$$\mathbf{f}_{[d]}(F) = \operatorname{graph} \Theta_k = \{\big(\mathbf{y}, \Theta_k(\mathbf{y})\big) : \mathbf{y} \in \bigtimes_{i=1}^{k} E_i\} \tag{1.9}$$

is a *k-dimensional* "surface" (parameterized by $E_1 \times \cdots \times E_k$) in $E_1 \times \cdots \times E_d$.

This suggests that to mark the interdependence of arbitrary $f_1, \ldots, f_d$, we need an index associated with $\mathbf{f}_{[d]}(F)$, which – properly defined – will effectively be its *dimension*.

2. Combinatorial dimension and fractional Cartesian products

2.1. **Definitions.** Given infinite sets $E_1, \ldots, E_d$, we consider an infinite subset

$$F \subset E_1 \times \cdots \times E_d,$$

and proceed to define its dimension relative to its ambient d-fold product $E_1 \times \cdots \times E_d$. With no *a priori* knowledge about structures in $E_1, \ldots, E_d$, all we can do is "count," and that is what we do: for every positive integer s, let

$$\Psi_F(s) = \max\{\#\big(F \cap (A_1 \times \cdots \times A_d)\big) : A_i \subset E_i,\ \#A_i = s,\ i \in [d]\}, \tag{2.1}$$

whence

$$s \leqslant \Psi_F(s) \leqslant s^d. \tag{2.2}$$

For example, if $F = \operatorname{graph} \Theta_k$, where Θ_k is given in (1.8), then

$$\Psi_F(s) = s^k, \quad s \in \mathbb{N}. \tag{2.3}$$

Now define

$$\dim F = \limsup_{s\to\infty} \frac{\log \Psi_F(s)}{\log s}. \tag{2.4}$$

From (2.2) and (2.4),

$$1 \leqslant \dim F \leqslant d, \tag{2.5}$$

and

$$\Psi_F(s) \sim s^{\dim F}, \tag{2.6}$$

by which we mean

$$\limsup_{s\to\infty} \frac{\Psi_F(s)}{s^\eta} = 0, \quad \eta > \dim F, \tag{2.7}$$

and

$$\liminf_{s\to\infty} \frac{\Psi_F(s)}{s^\eta} = \infty, \quad \eta < \dim F. \tag{2.8}$$

At the critical value, if

$$\limsup_{s\to\infty} \frac{\Psi_F(s)}{s^{\dim F}} > 0, \tag{2.9}$$

then $\dim F$ is said to be *exact*. Otherwise, if

$$\limsup_{s\to\infty} \frac{\Psi_F(s)}{s^{\dim F}} = 0, \tag{2.10}$$

then $\dim F$ is said to be *asymptotic*. If $\dim F = \alpha$, and

$$0 < \liminf_{s\to\infty} \frac{\Psi_F(s)}{s^\alpha} \leqslant \limsup_{s\to\infty} \frac{\Psi_F(s)}{s^\alpha} < \infty, \tag{2.11}$$

then F is said to be an α-*product*.

The exponent $\dim F$ marks the interdependence of coordinates of elements in F. Specifically, $\dim F$ registers the interdependence of the restrictions to F of the projections $\pi_i : F \to E_i,\ i \in [d]$,

$$\pi_i(\mathbf{y}) = y_i, \quad \mathbf{y} = (y_j)_{j\in[d]} \in F. \tag{2.12}$$

We refer to $\dim F$ as the *combinatorial dimension* of F. Indeed, the *combinatorial dimension* index provides a calibration of a basic counting principle:

(i) For every $\eta > \dim F$, there exists $K_\eta = K > 0$, so that for every $s \in \mathbb{N}$ and all s-sets $A_1 \subset E_1, \ldots, A_d \subset E_d$, if N is the number of samplings $x_1 \in A_1, \ldots, x_d \in A_d$, subject to the constraint that $(x_1, \ldots, x_d) \in F$, then

$$N \leqslant s^\eta. \tag{2.13}$$

(ii) For every $\eta < \dim F$, and all $L > 0$ (as large as we please), there exist s-sets $A_1 \subset E_1, \ldots, A_d \subset E_d$, such that if N is the number of samplings $x_1 \in A_1, \ldots, x_d \in A_d$, subject to the constraint that $(x_1, \ldots, x_d) \in F$, then

$$N > Ls^\eta. \tag{2.14}$$

2.2. **Fractional Cartesian Products.** Fix an integer $n > 1$, and let $D_1, \ldots, D_n$ be infinite sets. For $S \subset [n]$, define the projection

$$\pi_S : \bigtimes_{j=1}^{n} D_j \to \bigtimes_{j \in S} D_j \tag{2.15}$$

by

$$\pi_S(\mathbf{x}) = (x_j : j \in S), \quad \mathbf{x} = (x_1, \ldots, x_n) \in \bigtimes_{j=1}^{n} D_j. \tag{2.16}$$

Let $\mathcal{U} = \{S_1, \ldots, S_d\}$ be a cover of $[n]$; that is, $S_i \subset [n]$ for $i \in [d]$, and

$$\bigcup_{i=1}^{d} S_i = [n]. \tag{2.17}$$

Denote (for notational convenience)

$$\mathbf{D} = D_1 \times \cdots \times D_n, \tag{2.18}$$

and define the *fractional Cartesian product*

$$\mathbf{D}^{\mathcal{U}} := \big\{ \big(\pi_{S_1}(\mathbf{x}), \ldots, \pi_{S_d}(\mathbf{x})\big) : \mathbf{x} \in \mathbf{D} \big\}. \tag{2.19}$$

Specifically, in the setting of Section 1, we take

$$f_i = \pi_{S_i}, \quad i \in [d], \tag{2.20}$$

$$F = \mathbf{D}, \tag{2.21}$$

$$E_i = \bigtimes_{j \in S_i} D_j, \quad i \in [d], \tag{2.22}$$

and (cf. (1.2) and (1.3)) take $\mathbf{f}_{[d]}$ to be the function $\Pi_{\mathcal{U}} = \big(\pi_{S_1}, \ldots, \pi_{S_d}\big)$,

$$\Pi_{\mathcal{U}} : \mathbf{D} \to \big(\bigtimes_{j \in S_1} D_j \big) \times \cdots \times \big(\bigtimes_{j \in S_d} D_j \big). \tag{2.23}$$

Then,

$$\mathbf{D}^{\mathcal{U}} := \Pi_{\mathcal{U}}(\mathbf{D}) \subset \big(\bigtimes_{j \in S_1} D_j \big) \times \cdots \times \big(\bigtimes_{j \in S_d} D_j \big). \tag{2.24}$$

We refer to $\big(\bigtimes_{j \in S_1} D_j \big) \times \cdots \times \big(\bigtimes_{j \in S_d} D_j \big)$ as the *ambient product* of $\mathbf{D}^{\mathcal{U}}$. In this context, $\dim \mathbf{D}^{\mathcal{U}}$ marks the interdependence of the projections

$$\pi_S : D_1 \times \cdots \times D_n \to \bigtimes_{i \in S} D_i, \quad S \in \mathcal{U}. \tag{2.25}$$

Fractional Cartesian products provide natural examples of q-products for every rational $q = \frac{n}{k} \geqslant 1$. We state below two archetypal instances:

Example 2.1. *(Cf. [1].) If $\mathcal{U}$ is the cover of $[n]$ comprising all k-subsets of $[n]$, in which case*

$$\#\mathcal{U} = \binom{n}{k} = d, \tag{2.26}$$

then, there exist constants $\kappa_1 > 0$ and $\kappa_2 > 0$ such that for all integers $s \geqslant 1$,

$$\kappa_1 s^{\frac{n}{k}} \leqslant \Psi_{\mathbf{D}^{\mathcal{U}}}(s) \leqslant \kappa_2 s^{\frac{n}{k}}; \tag{2.27}$$

i.e., $\mathbf{D}^{\mathcal{U}}$ *is a* $\frac{n}{k}$*-product, whose ambient product is* $\binom{n}{k}$*-dimensional.*

Example 2.2. *(See [3].) Let* $\mathcal{U}$ *be a cover of* $[n]$ *comprising* k*-subsets of* $[n]$*, such that for every* $j \in [n]$*,*

$$\#\{S \in \mathcal{U} : j \in S\} = k, \tag{2.28}$$

whence

$$\#\mathcal{U} = n = d. \tag{2.29}$$

Then, there exist constants $\kappa_1 > 0$ *and* $\kappa_2 > 0$ *(different from the constants in (2.27)) such that for all integers* $s \geqslant 1$*,*

$$\kappa_1 s^{\frac{n}{k}} \leqslant \Psi_{\mathbf{D}^{\mathcal{U}}}(s) \leqslant \kappa_2 s^{\frac{n}{k}}; \tag{2.30}$$

i.e., $\mathbf{D}^{\mathcal{U}}$ *is a* $\frac{n}{k}$*-product, whose ambient product is* n*-dimensional.*

These two examples are instances – in fact, precursors – of a general result in [14], that the combinatorial dimension of a fractional Cartesian product is the solution to a linear programming problem:

Theorem 2.1. *Let* $\mathcal{U}$ *be a cover of* $[n]$*, and let*

$$\alpha = \alpha(\mathcal{U}) = \max\Big\{v_1 + \cdots + v_n : \sum_{j \in S} v_j \leqslant 1,\ S \in \mathcal{U};\ v_i \geqslant 0,\ i \in [n]\Big\}. \tag{2.31}$$

If $D_1, \ldots, D_n$ *are infinite sets, and* $\mathbf{D}^{\mathcal{U}}$ *is the fractional Cartesian product defined in (2.19), then* $\mathbf{D}^{\mathcal{U}}$ *is an* α*-product. Moreover (cf. (2.11)),*

$$\liminf_{s \to \infty} \frac{\Psi_{\mathbf{D}^{\mathcal{U}}}(s)}{s^\alpha} = \limsup_{s \to \infty} \frac{\Psi_{\mathbf{D}^{\mathcal{U}}}(s)}{s^\alpha} = 1. \tag{2.32}$$

2.3. **Deterministic designs vs. random sets.** Fractional Cartesian products provide explicit examples of $q-$dimensional sets for every rational $q \geqslant 1$. Notably, if $q = \frac{n}{k}$, where n and k are relatively prime, then the ambient product of an $\frac{n}{k}$-dimensional fractional Cartesian product must be at least n-dimensional. (In this sense, Example 2.2 is optimal.) Settling an obvious question, for all integers $d > 1$ and arbitrary $\alpha \in [1, d]$, there indeed exist α-(combinatorial) dimensional subsets in ambient d-dimensional products. However, except for finitely many $\alpha \in [1, d]$, hitherto all such examples have been randomly produced; see [2], [13], [8]. An open question remains: how to construct deterministically in a given ambient d-fold Cartesian product $\alpha-$ dimensional sets for arbitrary $\alpha \in [1, d]$.

3. Interdependence of infinitely many functions

3.1. **The setup.** Taking up the infinite-dimensional case ($d = \infty$), we now consider the interdependence of countably many functions

$$\chi_j : \Omega \to E_j, \quad j \in \mathbb{N}.$$

As before, we make no *a priori* assumptions about structures in Ω and E_j, but assume this time that the respective ranges are finite. We consider the simplest case

$$\# E_j = 2, \quad j \in \mathbb{N},$$

and may as well (for reasons that will soon become apparent) let

$$E_j = \{-1, +1\}, \quad j \in \mathbb{N}.$$

(For example, think of a "walk" with countably many steps, to the right or to the left, with no *a priori* assumptions or knowledge about their interdependence.)

As in the case $d < \infty$, we agree (speaking heuristically) that *interdependence* here means a link between evaluations of χ_i for $i \in T$, $T \subset \mathbb{N}$, and evaluations of χ_i for $i \in S$, $S \subset \mathbb{N}\backslash T$. As in the case $d < \infty$, *independence* – absence of any links – simply means: for every choice of signs $\epsilon_j = \pm 1$, $j \in \mathbb{N}$, there exists $t \in \Omega$, such that $\chi_j(t) = \epsilon_j$, $j \in \mathbb{N}$. (Cf. *functional independence*, as per Definition 1.1.) To be sure, in this instance we obtain *no* information about evaluations of χ_i for $i \in T$ from evaluations of χ_i for $i \in S \subset \mathbb{N}\backslash T$. As before, to gauge *interdpendence*, we consider the function (cf. (1.2), (1.3))

$$\mathbf{X} = (\chi_1, \ldots, \chi_n, \ldots) : \Omega \to \{-1, +1\}^{\mathbb{N}}, \tag{3.1}$$

its image in $\{-1, 1\}^{\mathbb{N}}$

$$\mathbf{X}(\Omega) = \big\{ \big(\chi_1(t), \ldots, \chi_n(t), \ldots\big) : \ t \in \Omega \big\} \subset \{-1, +1\}^{\mathbb{N}}, \tag{3.2}$$

and then posit that to gauge the interdependence of the χ_j, we need to assess the extent to which $\mathbf{X}(\Omega)$ differs from $\{-1, 1\}^{\mathbb{N}}$.

3.2. **A combinatorial measurement: the exponent ρ_X.** To facilitate discussions (here, and later in the sequel), denote $X = \{\chi_j : j \in \mathbb{N}\}$. (We distinguish between the set of functions X, and the function $\mathbf{X}$ defined in (3.1).) For $S \subset X$, define the function

$$\mathbf{X}_S : \Omega \to \{-1, +1\}^S$$

by

$$\mathbf{X}_S(t) = \big(\chi(t) : \chi \in S\big), \quad t \in \Omega,$$

and consider its image

$$\mathbf{X}_S(\Omega) := \big\{ \mathbf{X}_S(t) : t \in \Omega \big\} \subset \{-1, +1\}^S. \tag{3.3}$$

Let

$$\Phi_X(s) = \min\{\# \mathbf{X}_S(\Omega) : S \subset X, \ \# S = s\}. \tag{3.4}$$

We note

$$2 \leqslant \Phi_X(s) \leqslant 2^s, \quad s \in \mathbb{N}, \tag{3.5}$$

and then, to make precise the notion that the "closer" $\Phi_X(s)$ is to 2^s the "lesser" the interdependence in X (cf. Sect. 1), we define

$$\rho_X = \liminf_{s\to\infty} \frac{\log\log \Phi_X(s)}{\log s}; \tag{3.6}$$

to wit, larger ρ_X conveys less interdependence. From (3.5) and (3.6),

$$0 \leqslant \rho_X \leqslant 1, \tag{3.7}$$

and

$$\Phi_X(s) \sim 2^{s^{\rho_X}}, \tag{3.8}$$

which means

$$\liminf_{s\to\infty} \frac{\Phi_X(s)}{2^{s^\eta}} = \infty, \quad \eta < \rho_X, \tag{3.9}$$

and

$$\limsup_{s\to\infty} \frac{\Phi_X(s)}{2^{s^\eta}} = 0, \quad \eta > \rho_X. \tag{3.10}$$

3.3. **A functional-analytic measurement: p-variation.** First suppose that X is functionally independent (according to Definition 1.1, with $d = \infty$). In this case, if S is a finite subset of X, then for every choice of signs $\epsilon_\chi = \pm 1,\ \chi \in S$,

$$\| \sum_{\chi\in S} \epsilon_\chi \chi \|_\infty = \#S \tag{3.11}$$

($\|\cdot\|_\infty$ = supremum-norm over Ω). This is easy to verify: simply solve for $t \in \Omega$ such that

$$\chi(t) = \epsilon_\chi,\ \chi \in S, \tag{3.12}$$

which is possible by functional independence. In general, however,

$$\textit{maximizing} \quad \Big| \sum_{\chi\in S} \epsilon_\chi \chi(t) \Big| \quad \textit{over} \quad t \in \Omega \tag{3.13}$$

(left side of (3.11)) is constrained by the interdependence of the $\chi \in S$, which impedes the simultaneous solution of *all* equations in (3.12). The idea then is to assess interdependence via its constraining effect on (3.13). To this end, we use the l^p-norm, $p \geqslant 1$ (*p-variation*). Specifically, we use the notion that increasing interdependence – making a solution of (3.12) less feasible – decreases the maximum obtained in (3.13), and effectively increases the "smallest" $p \in [1, \infty)$ such that

$$\| \sum_{\chi\in S} \epsilon_\chi \chi \|_\infty \geqslant \Big(\sum_{\chi\in S} 1^p \Big)^{\frac{1}{p}} = \big(\#S\big)^{\frac{1}{p}}, \quad \text{finite } S \subset X, \quad (\epsilon_\chi) \in \{-1,1\}^S. \tag{3.14}$$

(These heuristics are supported by examples, and relations between the various calibrations; see Section 4.)

To make matters precise, define

$$\phi_X(s) = \min\{\|\sum_{\chi\in S}\epsilon_\chi\chi\|_\infty : S\subset X,\ |S| = s,\ (\epsilon_\chi)\in\{-1,1\}^S\},\tag{3.15}$$

and

$$\sigma_X = \limsup_{s\to\infty}\frac{\log s}{\log\phi_X(s)}.\tag{3.16}$$

The exponent σ_X is the "smallest" p that "works" in (3.14). We write

$$\phi_X(s)\sim s^{\frac{1}{\sigma_X}},\tag{3.17}$$

by which we mean (as usual)

$$\liminf_{s\to\infty}\frac{\phi_X(s)}{s^{\frac{1}{\eta}}} = \infty,\quad \eta > \sigma_X,\tag{3.18}$$

and

$$\limsup_{s\to\infty}\frac{\phi_X(s)}{s^{\frac{1}{\eta}}} = 0,\quad \eta < \sigma_X.\tag{3.19}$$

(Cf. (2.7), (2.8), (3.9), (3.10).) The index σ_X is *exact* if

$$\liminf_{s\to\infty}\frac{\phi_X(s)}{s^{\frac{1}{\sigma_X}}} > 0,\tag{3.20}$$

and *asymtotic* if

$$\liminf_{s\to\infty}\frac{\phi_X(s)}{s^{\frac{1}{\sigma_X}}} = 0.\tag{3.21}$$

(Cf. (2.9) and (2.10).) Because

$$\|\sum_{\chi\in S}\epsilon_\chi\chi\|_\infty \leqslant \#S,\quad (\epsilon_\chi)\in\{-1,1\}^S$$

for all finite $S\subset X$ (obviously!), we have

$$\sigma_X \geqslant 1.\tag{3.22}$$

Moreover, if we suppose that any two functions in X are functionally independent, then we deduce (via an exercise)

$$1\leqslant\sigma_X\leqslant 2.\tag{3.23}$$

Indeed, in a context of harmonic analysis, if X is an infinite spectral set, then σ_X is its *Sidon exponent*, and $\sigma_X\in[1,2]$. (See Section 4, and Section 5.3.)

3.4. **A statistical measurement: tail-probability estimates.** Next we measure interdependence via a "statistical" notion, that with less interdependence in X, cancellations are more likely in

$$\sum_{\chi \in S} \chi, \quad \text{finite } S \subset X. \tag{3.24}$$

To make this precise, we let $\mathbf{X}_S(\Omega)$ (finite $S \subset X$) be a uniform probability space, and denote the uniform probability measure on it by $\mathbb{P}_S$; i.e., ,

$$\mathbb{P}_S(\{\mathbf{y}\}) = \frac{1}{\#\mathbf{X}_S(\Omega)}, \quad \mathbf{y} \in \mathbf{X}_S(\Omega). \tag{3.25}$$

Consider the functions in S as random variables on $(\mathbf{X}_S(\Omega), \mathbb{P}_S)$, where

$$\chi(\mathbf{y}) = \mathbf{y}(\chi), \quad \mathbf{y} \in \mathbf{X}_S(\Omega) \subset \{-1+1\}^S. \tag{3.26}$$

(See (3.3).) We can now assess the likelihood of cancellations in (3.24) via estimates on tail-probabilities

$$\mathbb{P}_S\big(\big|\sum_{\chi \in S} \chi\big| \geqslant u\big), \quad \text{finite } S \subset X, \quad u > 0, \tag{3.27}$$

which we then use to calibrate interdependence in X.

We note that if the $\chi \in X$ are functionally independent (as functions on Ω), then every finite $S \subset X$ is statistically independent (as random variables on $(\mathbf{X}_S(\Omega), \mathbb{P}_S)$), and therefore by the classical Khintchin inequalities (e.g., see [15]),

$$\mathbb{P}_S\big(\frac{1}{\sqrt{\#S}}\big|\sum_{\chi \in S} \chi\big| \geqslant u\big) \leqslant \exp(-u^2), \quad \text{finite } S \subset X, \quad u > 0. \tag{3.28}$$

The "sub-Gaussian" estimates in (3.28) are indeed optimal in the case of independent X (e.g., cf. [11]), and will effectively mark the left-end point of a scale of interdependence.

Locating the right-end point of the scale is somewhat arbitrary, and is in fact tied to a long-standing open question known as the "$\Lambda(2)$-set problem"; e.g., see Section 5.1. In our discussion here, we assume that X at the very least has the property that every finite $S \subset X$ is orthogonal in $L^2\big(\mathbf{X}_S(\Omega), \mathbb{P}_S\big)$. (Ruling out "too high" a level of functional dependence in X, this assumption postulates a minimal amount of "randomness," and – as we see in the next section – still makes the mathematics here interesting...) With this assumption and Markov's inequality, we are guaranteed that *always*, for all finite $S \subset X$,

$$\mathbb{P}_S\big(\frac{1}{\sqrt{\#S}}\big|\sum_{\chi \in S} \chi\big| \geqslant u\big) \leqslant \frac{1}{u^2}, \quad u > 0. \tag{3.29}$$

We define for $u > 0$,

$$T_X(u) = \sup\big\{\mathbb{P}_S\big(\frac{1}{\sqrt{\#S}}\big|\sum_{\chi \in S} \chi\big| \geqslant u\big) : \text{finite } S \subset X\big\}. \tag{3.30}$$

From (3.29) and the optimality of (3.28), we obtain for some $\kappa > 0$

$$\exp(-\kappa u^2) \leqslant T_X(u) \leqslant \frac{1}{u^2}, \quad u > 0. \tag{3.31}$$

We now posit that the "level" of interdependence in X corresponds to the "location" of T_X on the "interval," whose respective "end-points" are $\exp(-\kappa u^2)$ and $\frac{1}{u^2}$. Two distinct scales are obtained, effectively marked by $\exp(-\kappa u^\alpha)$ ($\alpha \in [0,2]$), and by $\frac{1}{u^\xi}$ ($\xi \in [2,\infty]$). To calibrate these scales, we define

$$\delta_X = \liminf_{u\to\infty} \frac{\log\log T_X(u)^{-1}}{\log u}, \tag{3.32}$$

and

$$\lambda_X = \liminf_{u\to\infty} \frac{\log T_X(u)^{-1}}{\log u}. \tag{3.33}$$

By (3.31),

$$\delta_X \in [0,2], \tag{3.34}$$

and

$$\lambda_X \in [2,\infty], \tag{3.35}$$

where, on each scale, increasing index conveys decreasing interdependence. The two scales complement each other: the λ-scale is a resolution of the end-point $\delta_X = 0$ on the δ-scale, and the δ-scale is a resolution of the right end-point $\lambda_X = \infty$ on the λ-scale. If $\delta_X \in (0,2]$ or $\lambda_X \in [2,\infty)$, then (respectively)

$$T_X(u) \sim \exp(-u^{\delta_X}), \tag{3.36}$$

or

$$T_X(u) \sim \frac{1}{u^{-\lambda_X}}, \tag{3.37}$$

each of which means (respectively)

$$\limsup_{u\to\infty} \frac{T_X(u)}{\exp(-u^\eta)} = 0, \quad \eta < \delta_X, \tag{3.38}$$

$$\liminf_{u\to\infty} \frac{T_X(u)}{\exp(-u^\eta)} = \infty, \quad \eta > \delta_X, \tag{3.39}$$

and

$$\limsup_{u\to\infty} \frac{T_X(u)}{u^{-\eta}} = 0, \quad \eta < \lambda_X, \tag{3.40}$$

$$\liminf_{u\to\infty} \frac{T_X(u)}{u^{-\eta}} = \infty, \quad \eta > \lambda_X. \tag{3.41}$$

If

$$\liminf_{u\to\infty} \frac{\log T_X(u)^{-1}}{u^{\delta_X}} > 0, \tag{3.42}$$

or

$$\limsup_{u\to\infty} \frac{T_X(u)}{u^{-\lambda_X}} < \infty, \tag{3.43}$$

then (respectively) δ_X and λ_X are said to be *exact*. Otherwise, if either

$$\liminf_{u\to\infty} \frac{\log T_X(u)^{-1}}{u^{\delta_X}} = 0 \tag{3.44}$$

or

$$\limsup_{u\to\infty} \frac{T_X(u)}{u^{-\lambda_X}} = \infty, \tag{3.45}$$

then the indices are said to be *asymptotic*.

4. Examples and conversion formulas

We have:

$$\dim F \in [1,\infty), \quad \rho_X \in [0,1], \quad \sigma_X \in [1,\infty), \quad \delta_X \in [0,2], \quad \lambda_X \in [2,\infty].$$

Question 4.1. *Can every point on each scale be realized as an evaluation of the respective index?*

We have already noted (in Section 2.3) an affirmative answer in the case of *combinatorial dimension.* But what about the other indices?

A second question concerns links between the indices. If indeed each index registers, separately, a degree of interdependence, then we should expect conversion formulas between them:

Question 4.2. *What relations exist between the indices?*

4.1. **The view through harmonic analysis.** We take the compact Abelian group $\Omega = \{-1,1\}^{\mathbb{N}}$, with coordinate-wise multiplication, the product topology, and the Haar measure $\mathbb{P}$ on it (infinite product of the uniform measure on $\{-1,+1\}$). Let $W = \widehat{\Omega}$ denote its group of characters (Walsh characters); let R denote the set of projections from Ω onto its independent coordinates (Rademacher characters), and let W_k denote the set of Walsh characters of order k (products of k distinct Rademacher characters). The Rademacher characters form an algebraically independent "basis" for the Walsh characters. Specifically,

$$W = \bigcup_{k=0}^{\infty} W_k \ ; \tag{4.1}$$

$W_0 = \{r_0\}$, where r_0 is the function that identically equals 1 on Ω.

We consider first the "finite-dimensional" case. Fix an integer $k \geqslant 2$, and let $X \subset W_k$ be infinite. The objective (as always) is to gauge the interdependence of elements in X.

I (*Combinatorial dimension*). By the algebraic independence of R, we naturally identify X with a subset $\tilde{X}$ of R^k. Specifically, let

$$\tilde{X} = \big\{(r_1, \ldots, r_k) \in R^k : r_1 \cdots r_k \in X\big\}, \tag{4.2}$$

and then (unambiguously) define $\dim F$ to be $\dim \tilde{X}$. In our present context, $\dim X$ registers the "amount of freedom" we have in choosing k distinct Rademacher characters such that their product is in X.

By results stated in Section 2.3, for every $\alpha \in [1, k]$, there exist $X \subset W_k$ such that $\dim X = \alpha$ exactly, as well as $X \subset W_k$ such that $\dim X = \alpha$ asymptotically.

II (*The index* ρ_X). Following the connections between the combinatorial measurements applied to $\mathbf{X}(\Omega)$ (as per Section 3.2) and the combinatorial measurements applied to $\tilde{X}$ (as per Section 2.3), we obtain

$$\rho_X = \frac{1}{\dim X}. \tag{4.3}$$

III (*The Sidon exponent* σ_X). Following results in [2], we have

$$\sigma_X = \frac{2}{1 + \frac{1}{\dim X}}\ , \tag{4.4}$$

and that σ_X is exact if and only if $\dim X$ is exact. This relation implies that for every $\alpha \in [1, 2]$, there exist $X \subset W_k$ such that $\sigma_X = \alpha$ (exactly, or asymptotically).

IV (*The index* δ_X). Following results in [2] (again), we have

$$\delta_X = \frac{2}{\dim X}\ , \tag{4.5}$$

and that δ_X is exact if and only if $\dim X$ is exact. Therefore, for every $\alpha \in [0, 2]$ there exist $X \subset W_k$ such that $\delta_X = \alpha$ (exactly, or asymptotically).

V (*The index* λ_X). Since $\lambda_X = \infty$ for all $X \subset W_k$, $k = 2, \ldots$, to answer Question 4.1 in the case of the λ-scale, one now considers $X \subset W$. The problem concerning a "continuous" realization of the λ-scale had been raised in [18] (using the terminology of $\Lambda(p)$-sets), and resisted solution for nearly three decades. It was solved in the affirmative in two landmark papers: in [16], and later by a different method in [19].

Regarding Question 4.2, I know of a link between combinatorial measurements and the λ-scale only for $\lambda_X = 2j + 2,\ j \in \mathbb{N}$. (E.g., see [18].) A precise relation between the λ-scale and a tractable "combinatorial" scale is an open(-ended) problem.

5. Some further questions

5.1. ***Exact vs. Asymptotic?*** If an index is *asymptotic*, then – at least in principle – it can be resolved on a finer scale. For example, there exist (random sets) $F \subset E_1 \times \cdots \times E_d$ such that

$$\limsup_{s\to\infty} \frac{\Psi_F(s)}{s^{\dim F}} = 0 \tag{5.1}$$

(i. e., $\dim F$ is *asymptotic*), while for some $\beta > 0$,

$$0 < \limsup_{s\to\infty} \frac{\Psi_F(s)}{s^{\dim F}(\log s)^{\beta}} < \infty. \tag{5.2}$$

Certainly in this particular case it is natural to compare $\Psi_F(s)$ to $s^{\eta}(\log s)^{\xi}$. In the setting of the previous section, if $X \subset W_k$, then the challenge is to pick out the "exact" calibrations based on $\Psi_{\tilde{X}}(s)$, $\phi_X(s)$, and $T_X(u)$, and derive relations between them. This problem was considered in [9] and [10], specifically *vis a vis* sharper resolutions of the dim-, σ-, and δ-scales.

For every $\lambda \in (2,\infty)$, there exist spectral sets $X \subset W$ (based on constructions in [16] and [19]) such that $\lambda_X = \lambda$ *exactly*, as well as $X \subset W$ such that $\lambda_X = \lambda$ *asymptotically*. The left-end point of the λ-scale is still a mystery, and its "resolution" is an open(-ended) problem. To wit, whereas $\lambda_W = 2$ *exactly*, the class of $X \subset W$ for which $\lambda_X = 2$ is not well understood. For example, the following question is open: are there $X \subset W$ such that $\lambda_X = 2$, and

$$\inf\left\{\mathbb{P}\Big(\frac{1}{\sqrt{\#S}}\Big|\sum_{\chi\in S}\chi\Big| \geqslant 1\Big) : \text{finite } S \subset X\right\} > 0? \tag{5.3}$$

This question is closely related to an open problem of long standing, stated first (so far that I know) in [18], and widely known (among harmonic analysts) as the "$\Lambda(2)$-set problem."

5.2. **General relations between indices?** An obvious question arises: do the conversion formulas in Section 4 hold in the framework of Section 3? In Section 3, if X is functionally independent, then we can view it as a system of Rademacher characters on $\Omega = \{-1,+1\}^{\mathbb{N}}$, in which instance all the relations are trivially verified. Generally, *every* countable collection X of $\{-1,+1\}$-valued functions can be naturally viewed as a countable collection of Borel measurable functions on the compact Abelian group Ω of Section 4 (via the observation that every countably generated sigma-field can be generated by a countable collection of Boolean independent sets [17]). However, while the question concerning conversion formulas can be rephrased in the language of Section 4, I do not know the answer even in a special case, which is of interest to harmonic analysts: if $X \subset W$ is infinite, then do the following relations hold (cf. (4.3), (4.4), and (4.5)):

$$\sigma_X = \frac{2}{1+\rho_X} \ ? \tag{5.4}$$

$$\delta_X = 2\rho_X \ ? \tag{5.5}$$

5.3. **How "large" is the σ-scale?** If any two elements of X are functionally independent, then $\sigma_X \leqslant 2$. (See (3.23).) For $\sigma \in (2, \infty)$, are there systems X of $\{-1, +1\}$-valued functions such that $\sigma_X = \sigma$?

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E-mail address: Ron Blei, blei@math.uconn.edu

Department of Mathematics, University of Connecticut, Storrs, CT 06269