Scattering and Sparse Partitions, and their Applications

Arnold Filtser

Columbia University

January 17, Simons A&G collaboration

Universal Steiner tree

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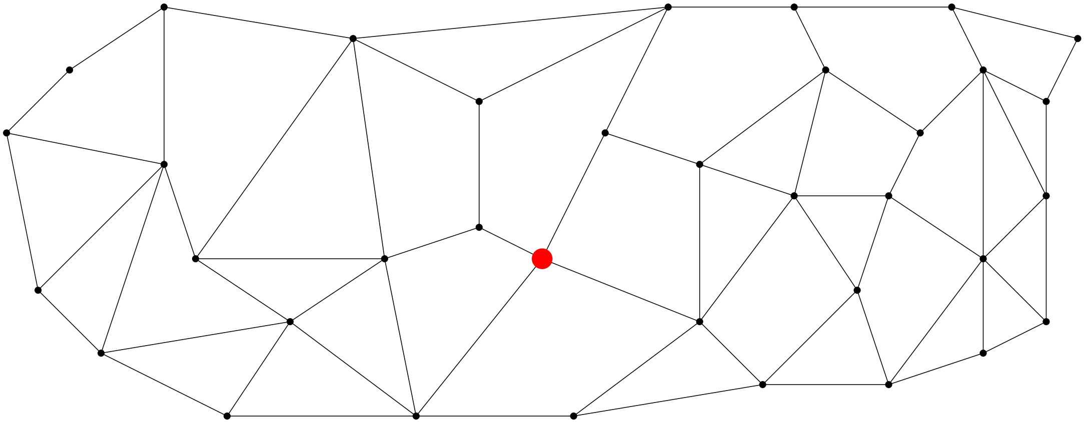

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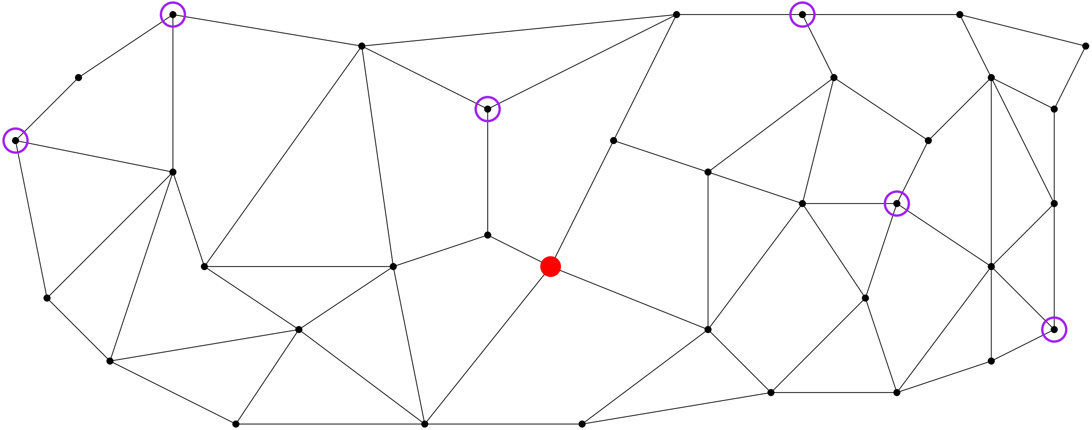

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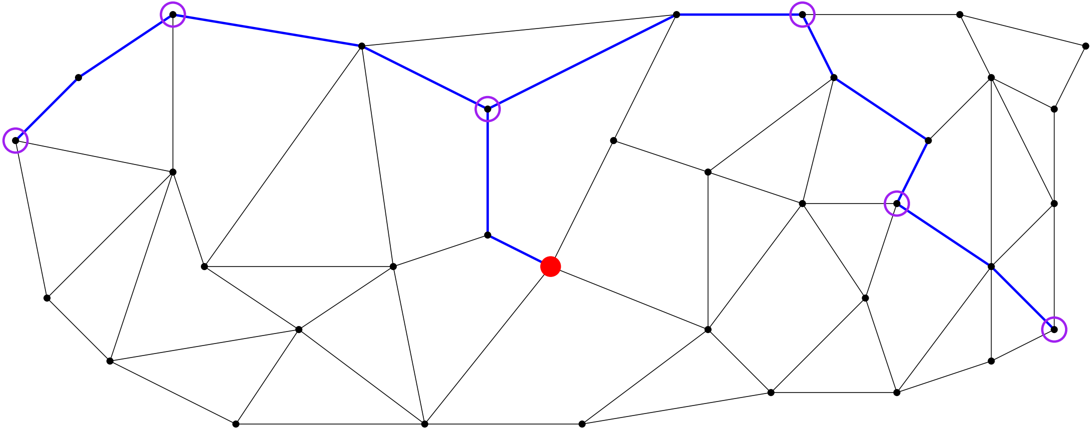

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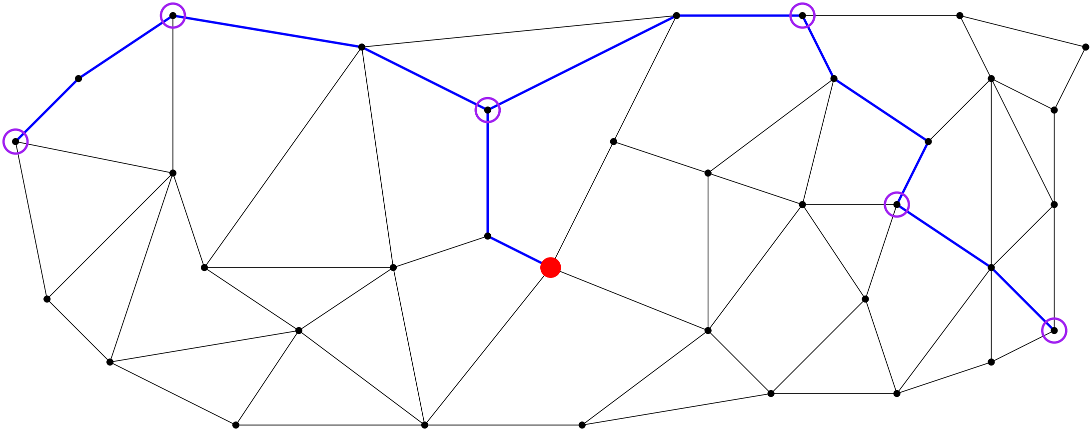

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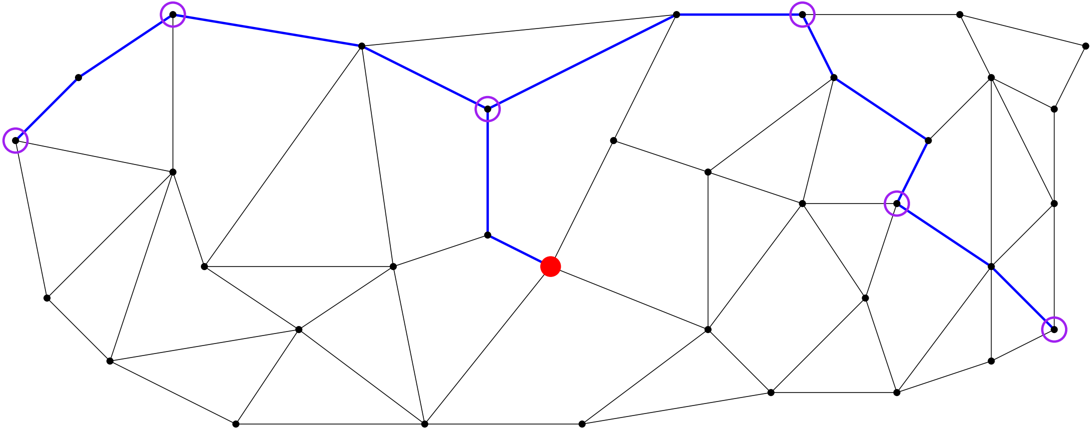

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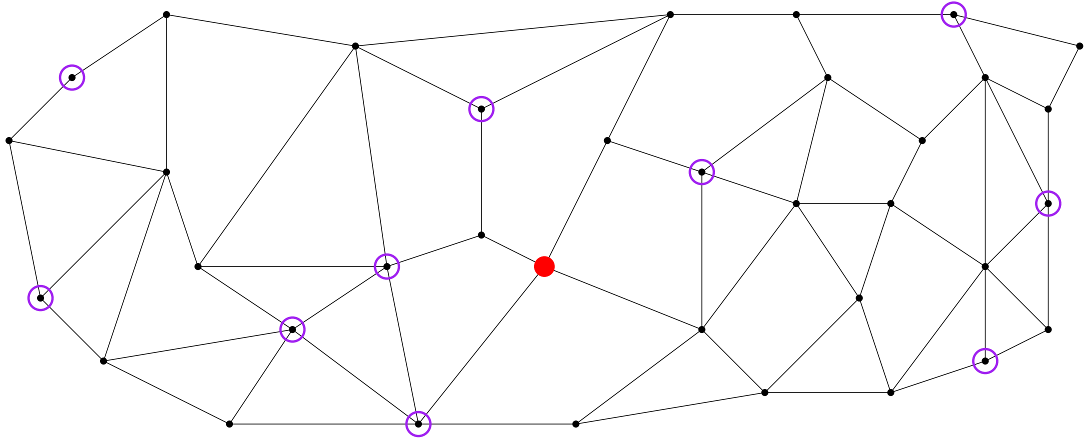

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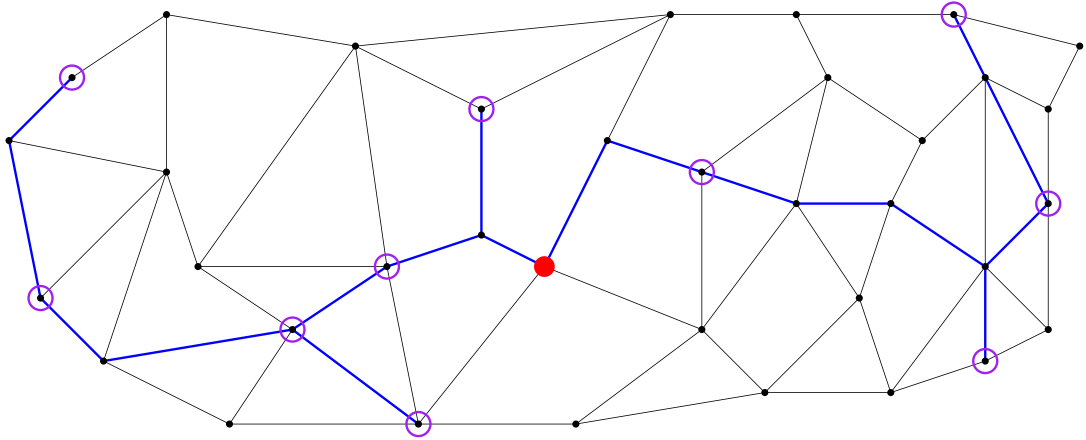

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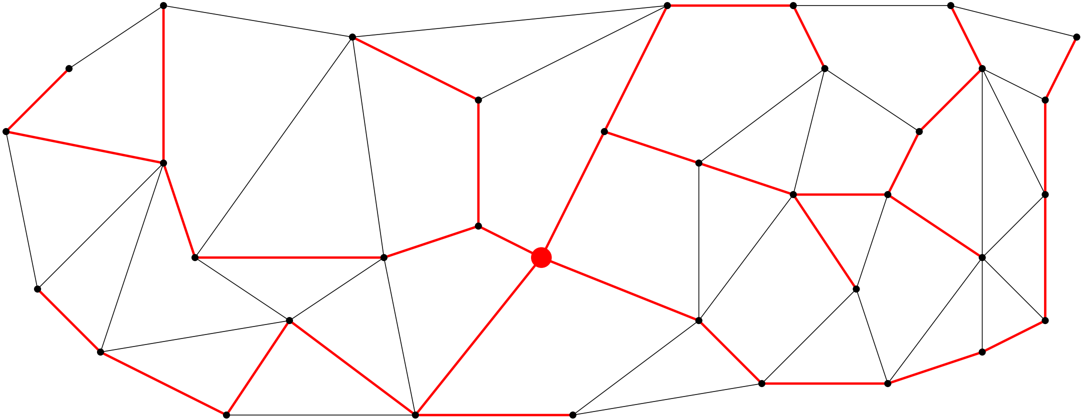

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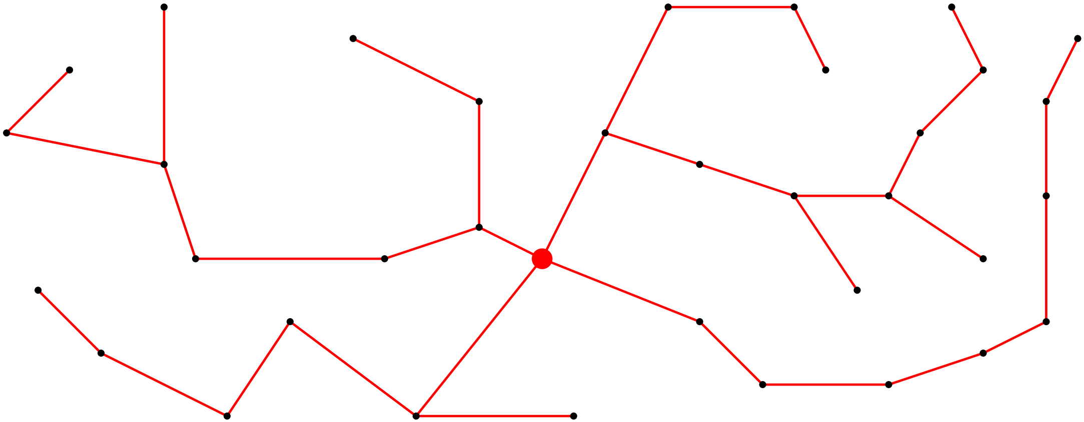

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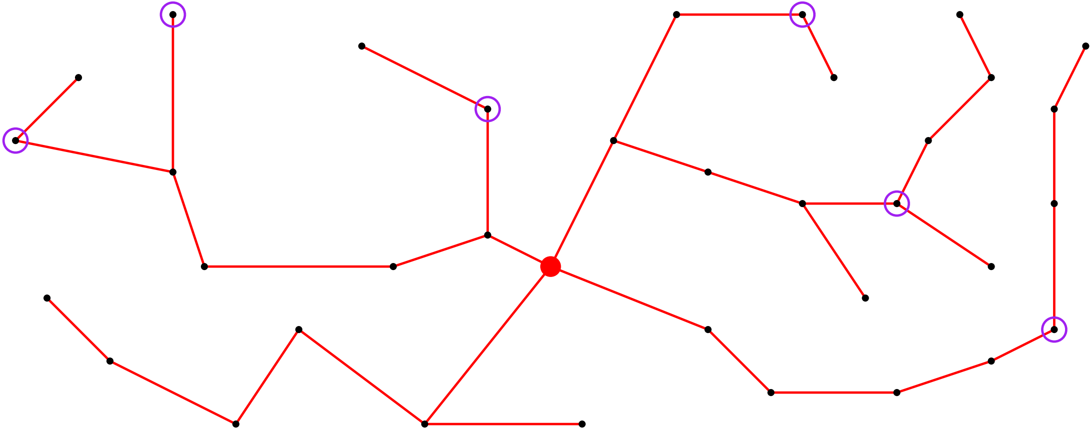

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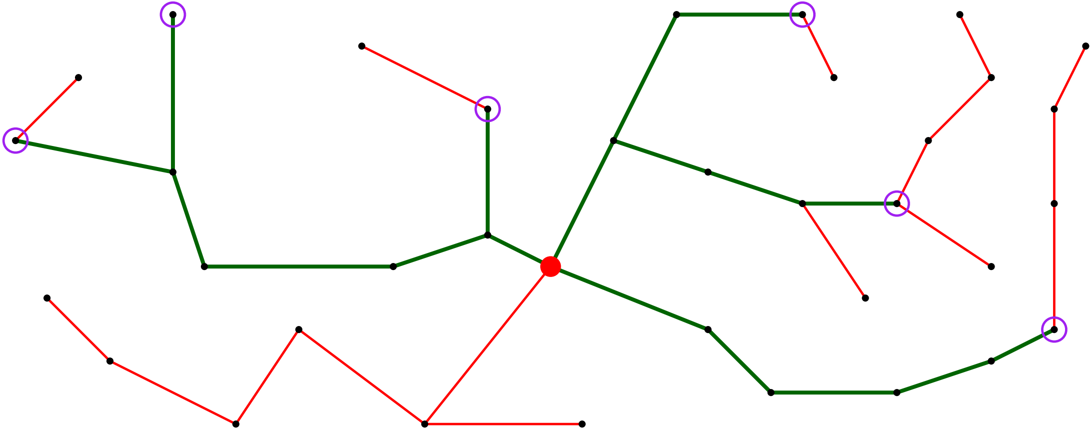

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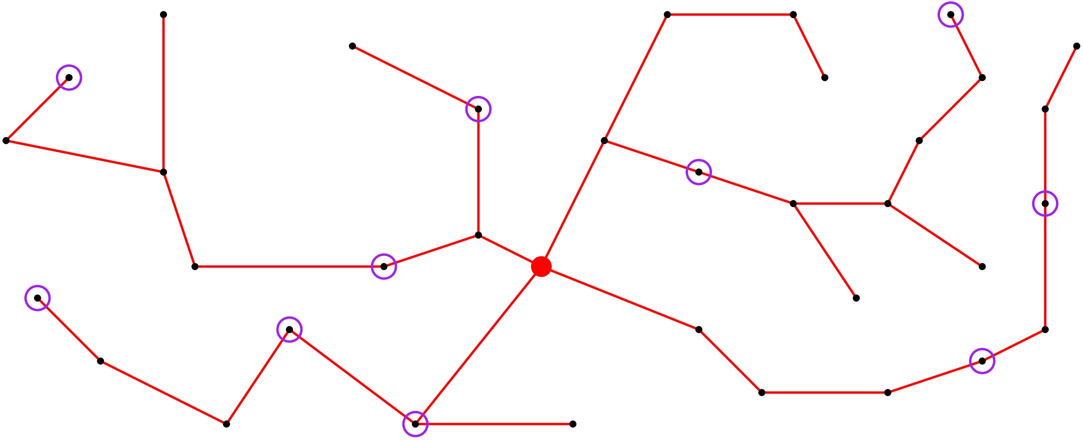

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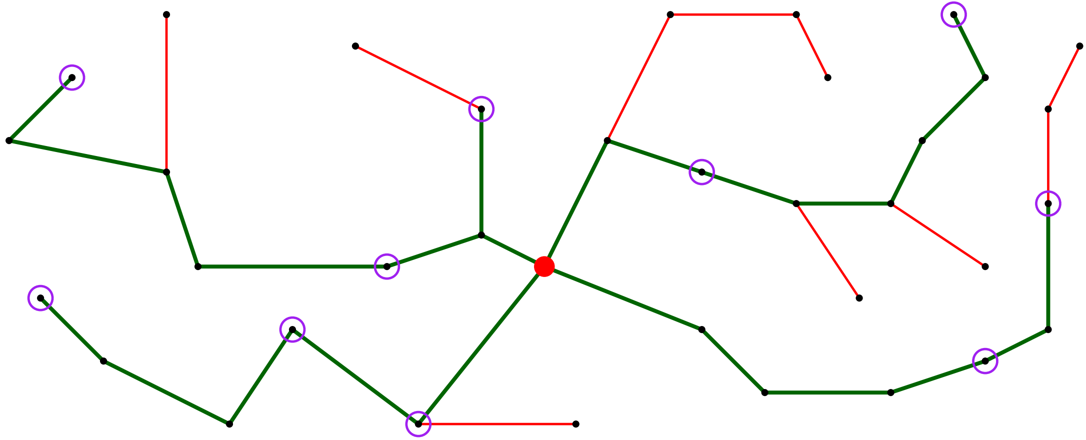

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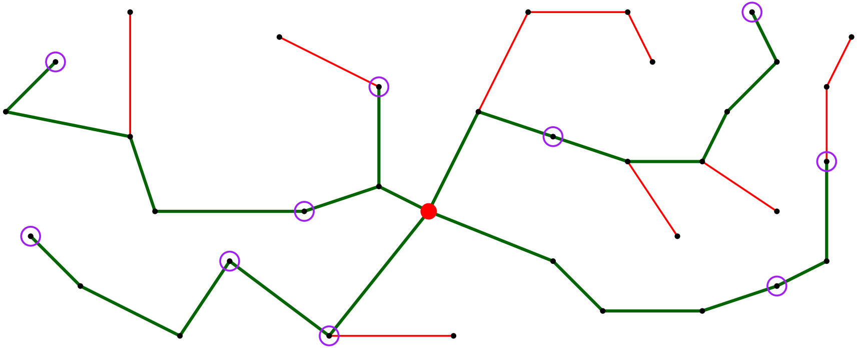

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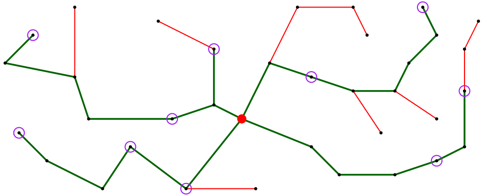

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Theorem ([Jia, Lin, Noubir, Rajaraman, Sundaram 05])

*Suppose G admits (σ, τ)-**sparse** partition scheme,*

$\Rightarrow$ *solution to the **UST** problem with stretch $O(\tau\sigma^2 \log_\tau n)$.*

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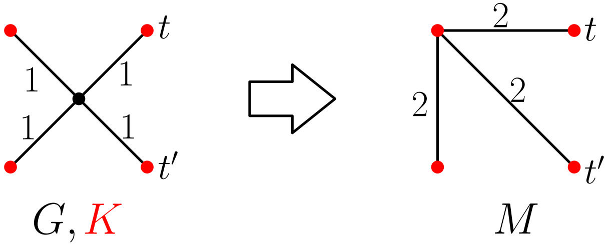

The distortion is: $\frac{d_M(t,t')}{d_G(t,t')} = \frac{4}{2} = 2$

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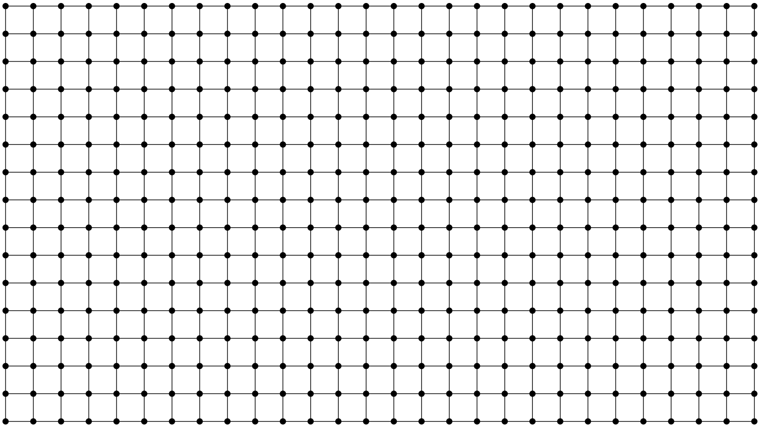
What about special graph families?

Theorem ([Fil 20])

*Suppose that every **induced subgraph** $G[A]$ of G admits (σ,τ)-scattering partition scheme, $\Rightarrow$ solution to the **SPR** problem with distortion $O(\tau^3\sigma^3)$.*

Sparse partitions

$\mathcal{P}$ is a (σ, τ, Δ)-**sparse partition** if:



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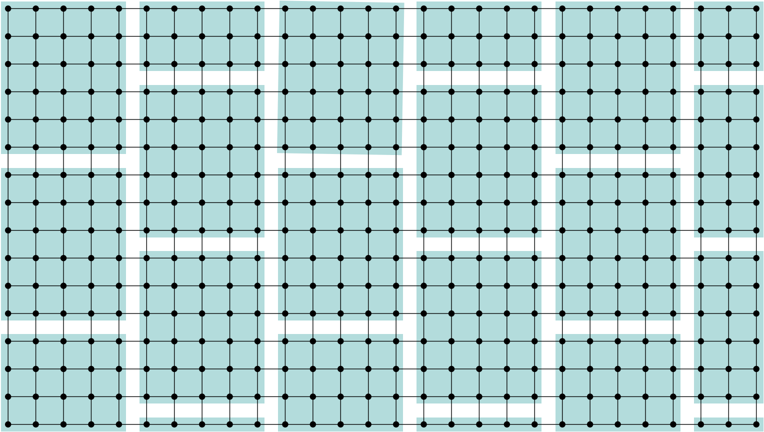

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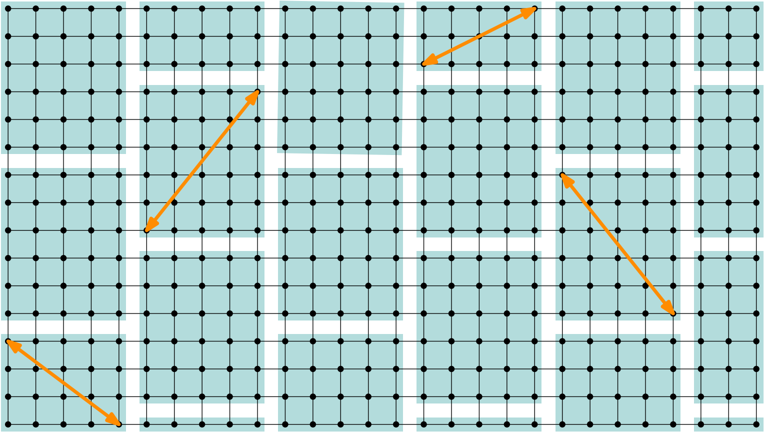

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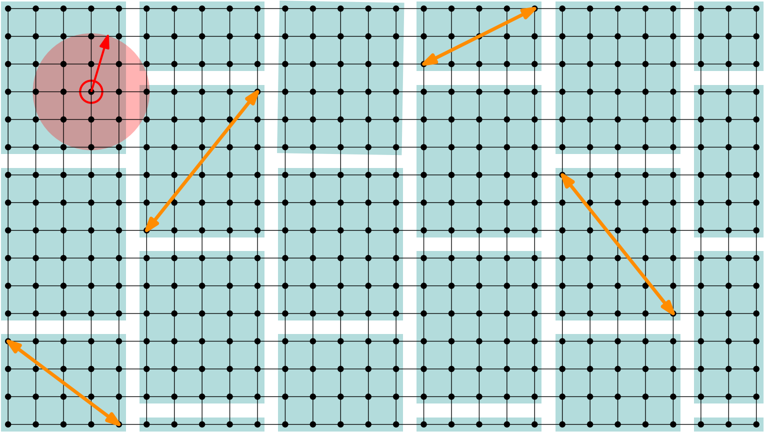

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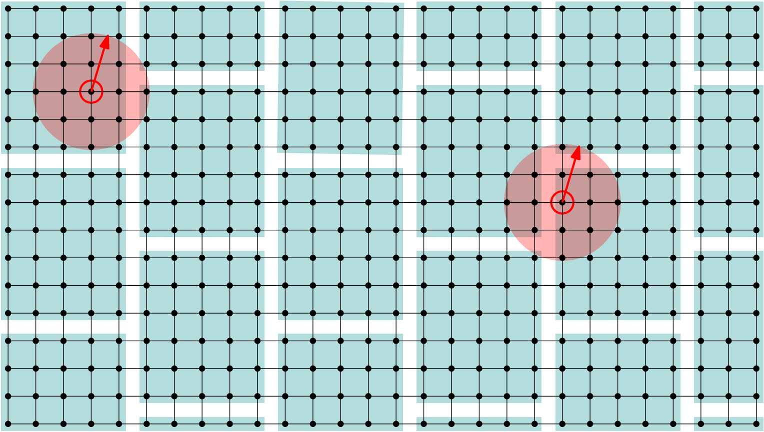

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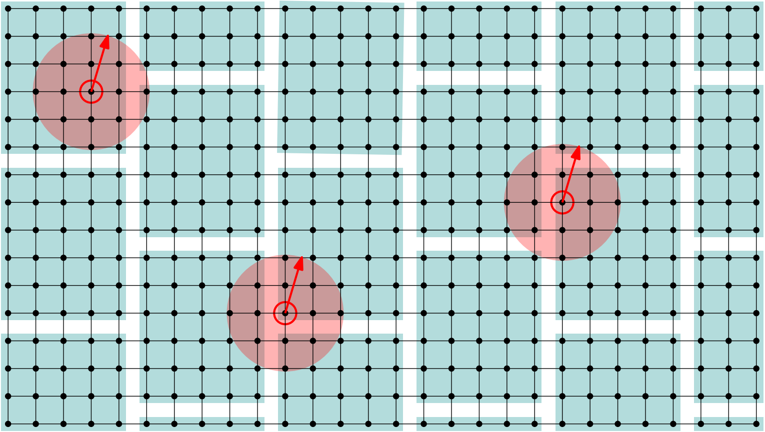

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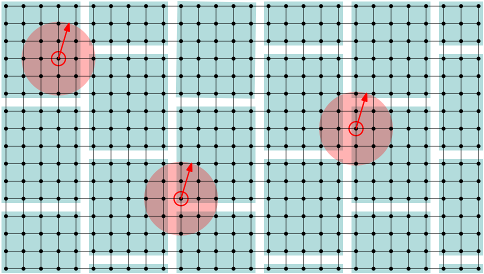

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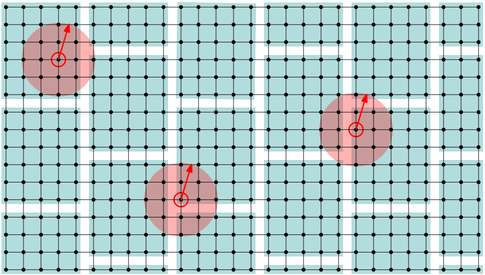

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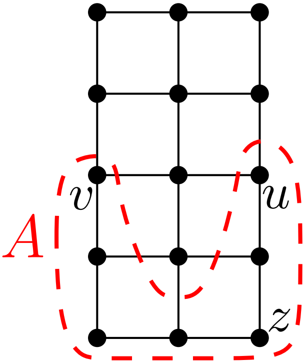

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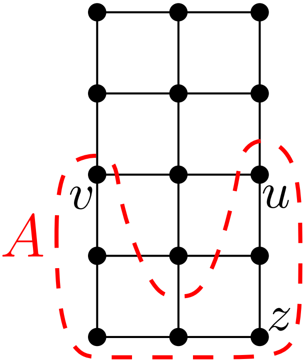

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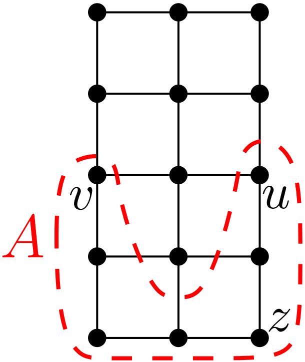

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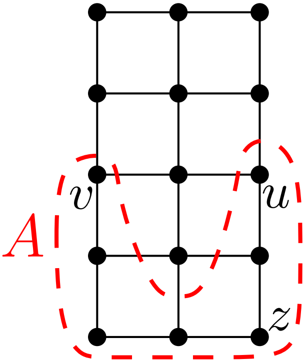

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Weak diameter of $A = 4$.
Strong diameter of $A = 6$.

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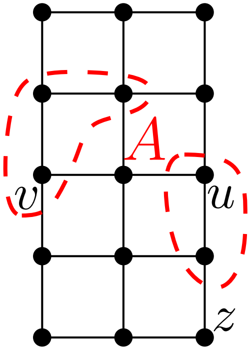

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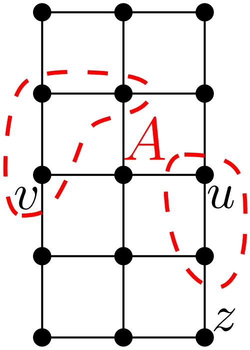

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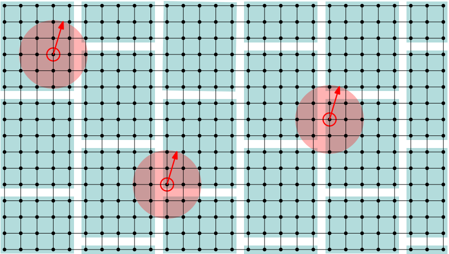

- The **strong/weak** diameter of each cluster $\leq \Delta$.
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(σ, τ)-**strong/weak** sparse partition scheme: $\exists$ (σ, τ, Δ)-**strong/weak** sparse partition for all $\Delta > 0$.

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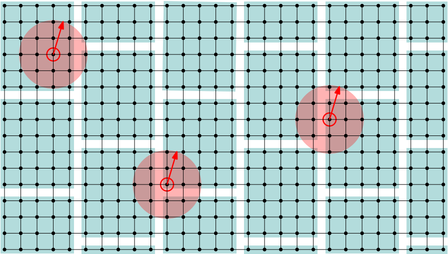

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[BDRRS 12]: subgraph solution using **hierarchy** of strong sparse partitions.

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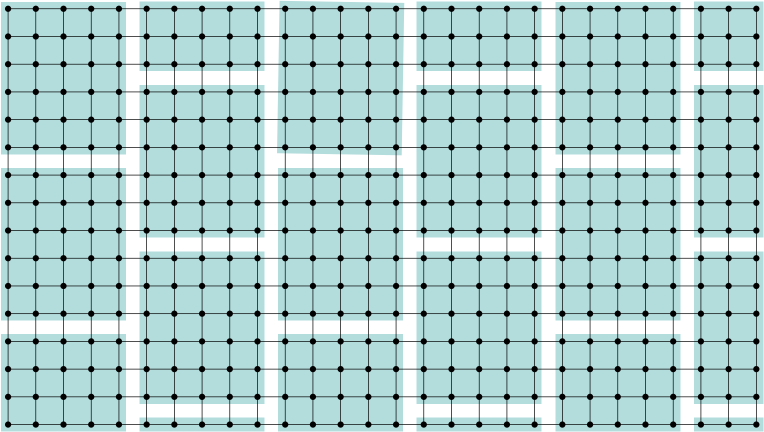

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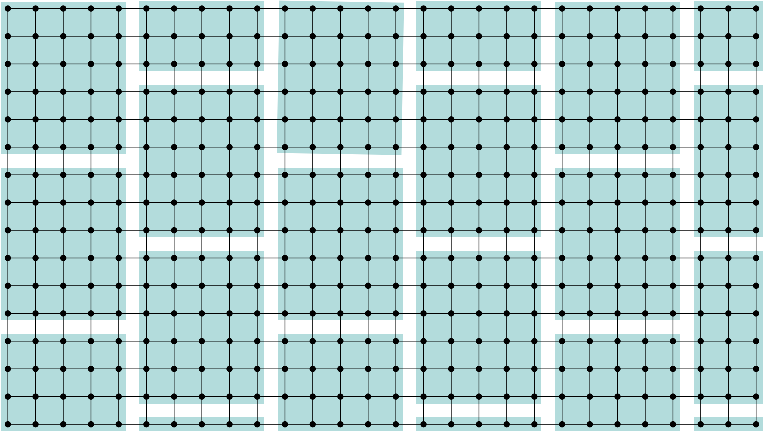

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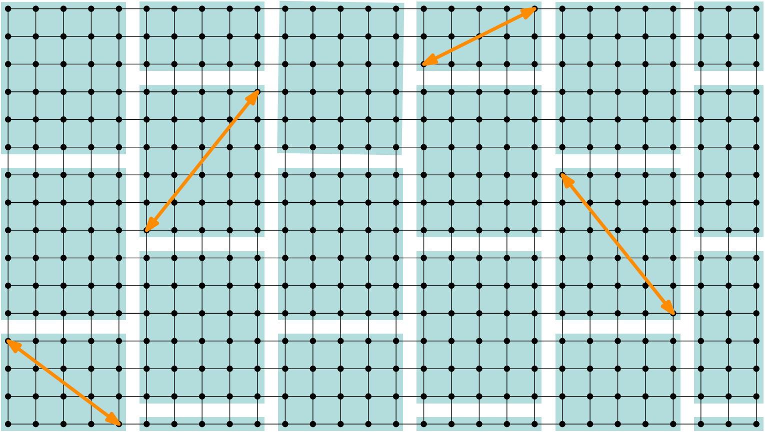

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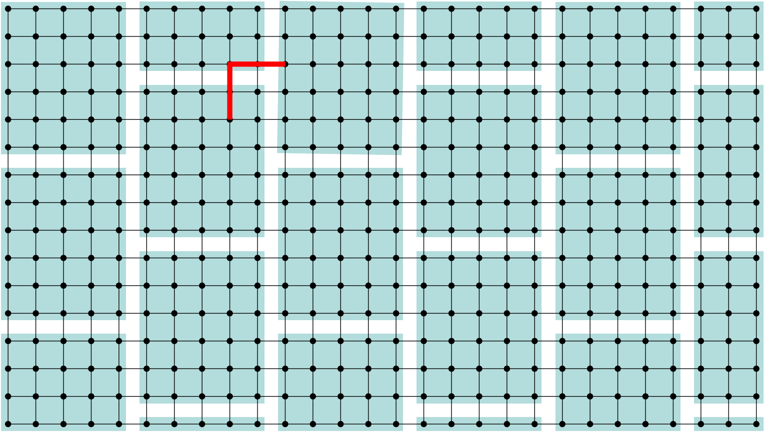

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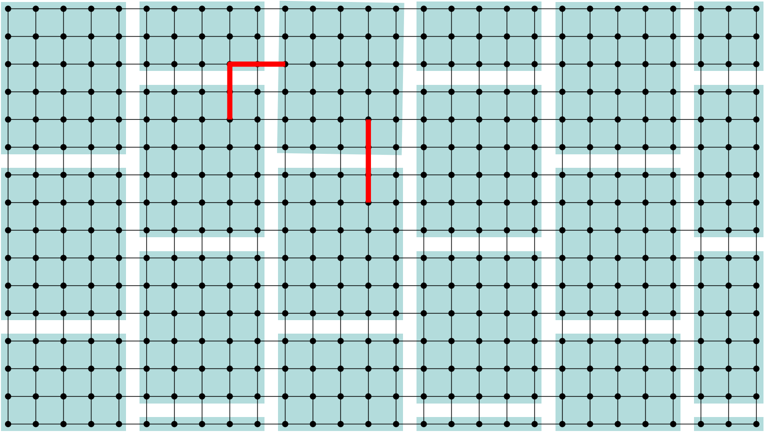
$\mathcal{P}$ is a (σ, τ, Δ)-**scattering partition** if:



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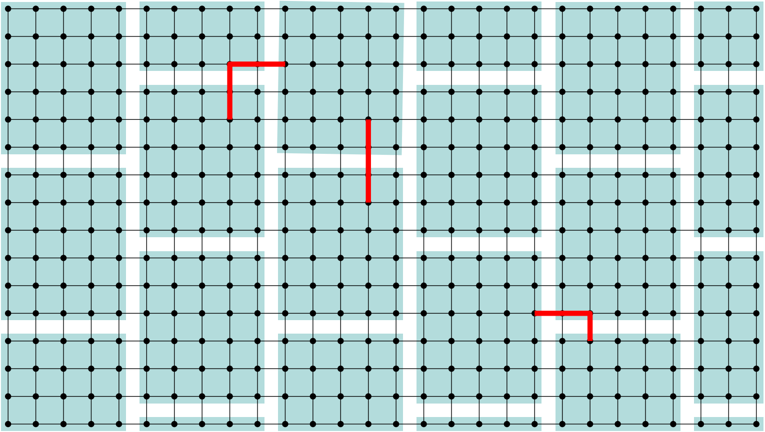

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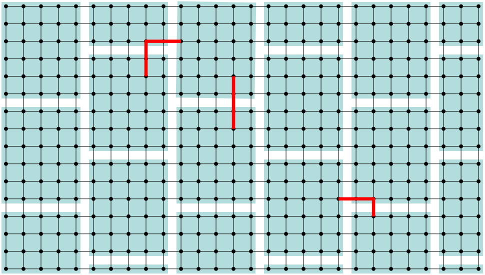

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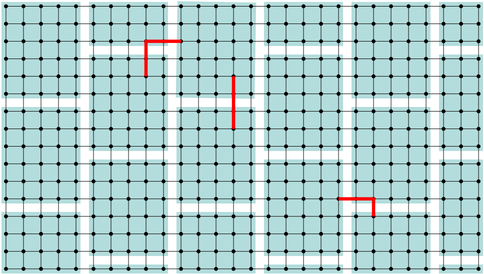

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(σ,τ)-**scattering partition scheme**: $\forall \Delta > 0 \quad \exists$ (σ,τ,Δ)-scattering partition.

Theorem ([Fil 20])

Suppose that every **induced subgraph** $G[A]$ *of* G *admits* (σ,τ)*-scattering partition scheme,* $\Rightarrow$ *solution to the* **SPR** *problem with distortion* $O(\tau^3\sigma^3)$.

Observations

$$(\sigma, \tau, \Delta)\text{-strong sparse} \quad \Rightarrow \quad (\sigma, \tau, \Delta)\text{-weak sparse} \;.$$

- Each cluster **strong** diameter $\leq \Delta$.
- Every ball of radius $\leq \frac{\Delta}{\sigma}$ intersects at most τ clusters.

- Each cluster **weak** diameter $\leq \Delta$.
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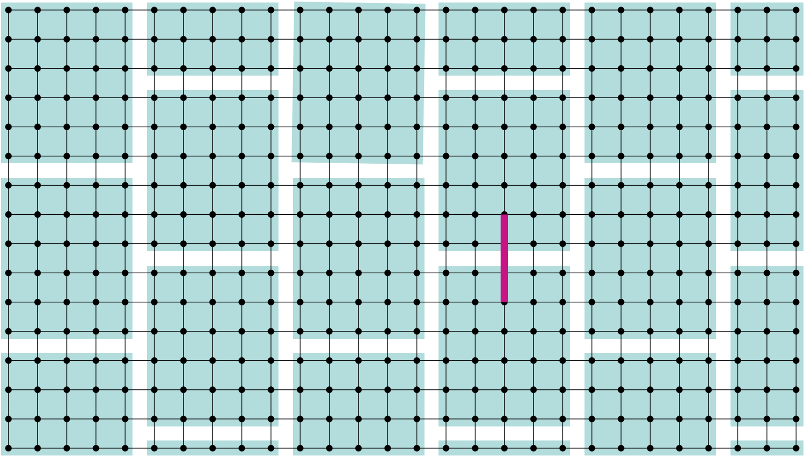

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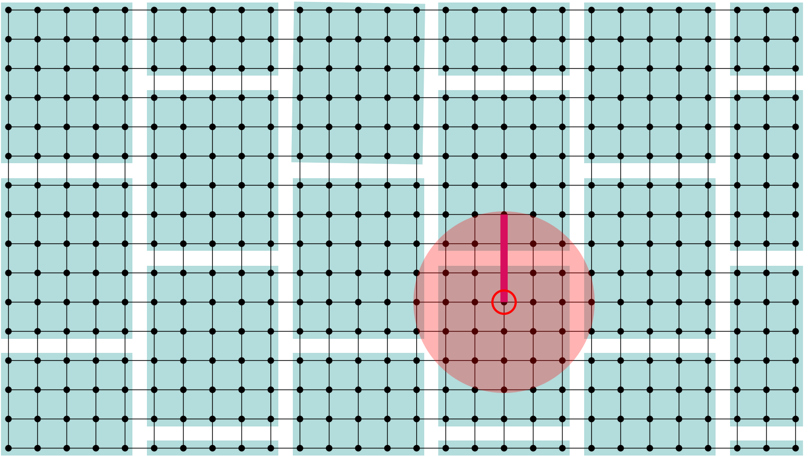
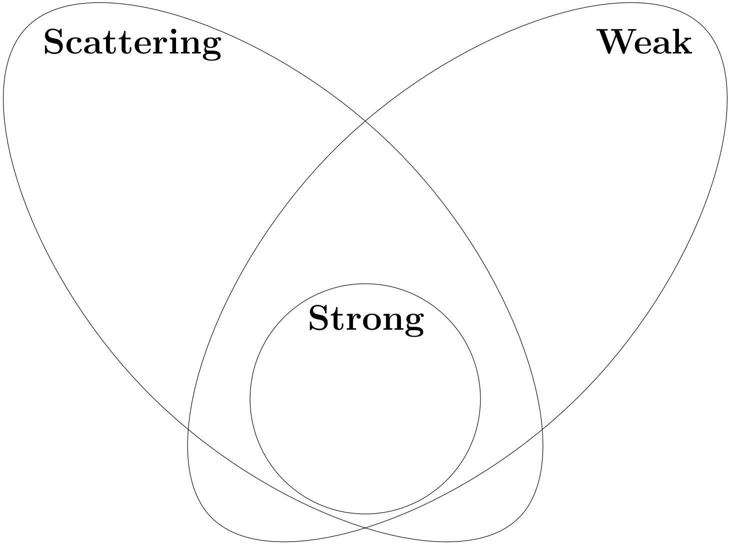
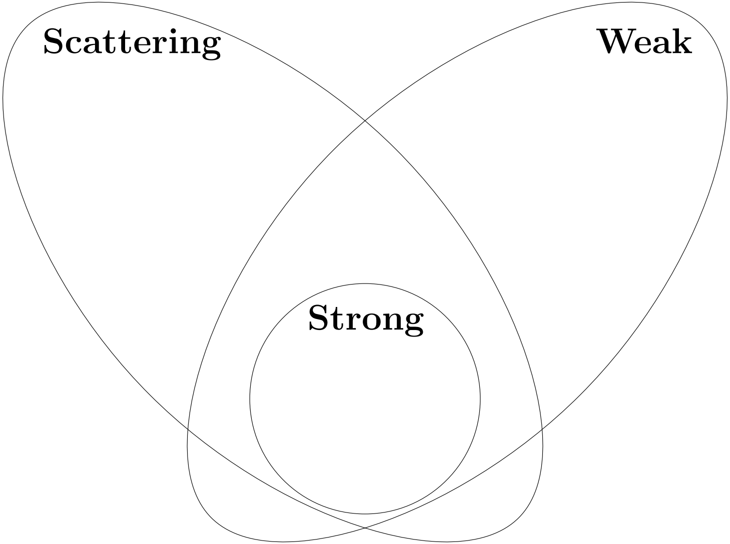
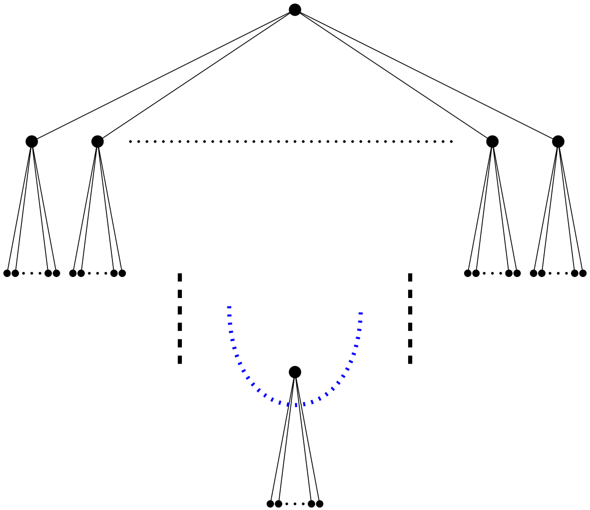



Scattering
Weak
Strong



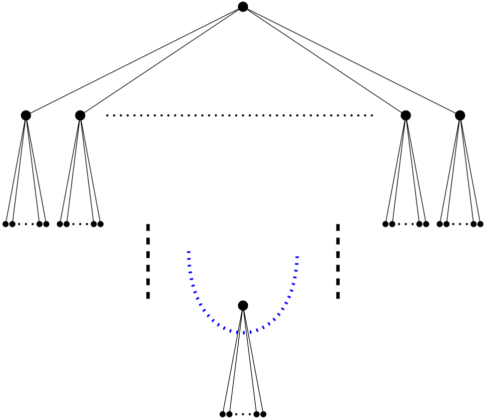
Trees?



Theorem ([**Fil** 20])

Suppose all n-vertex **trees** *admit a* (σ, τ)-**strong** *sparse partition scheme.*

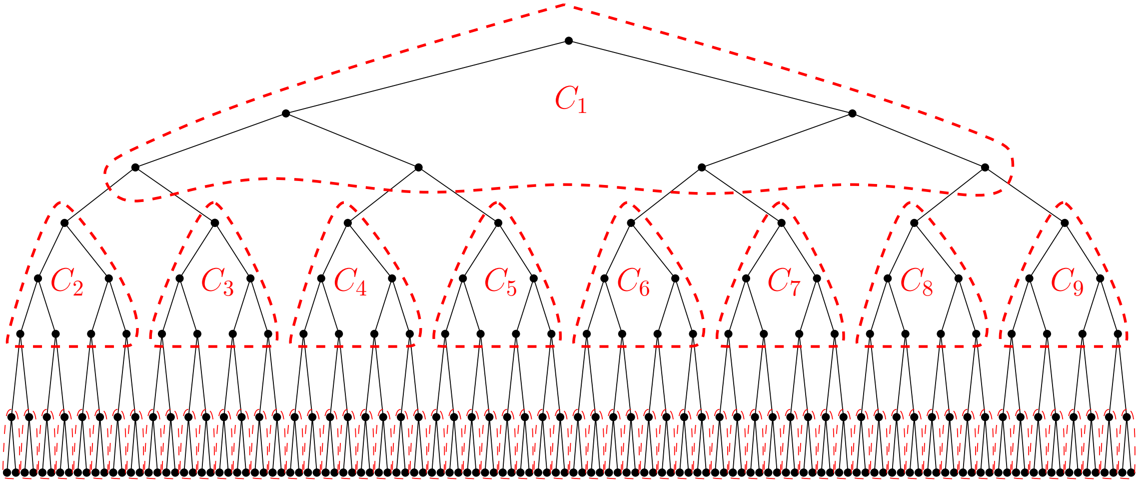
Then $\boldsymbol{\tau} \geq \frac{1}{3} \cdot n^{\frac{2}{\sigma+1}}$.



Corollary

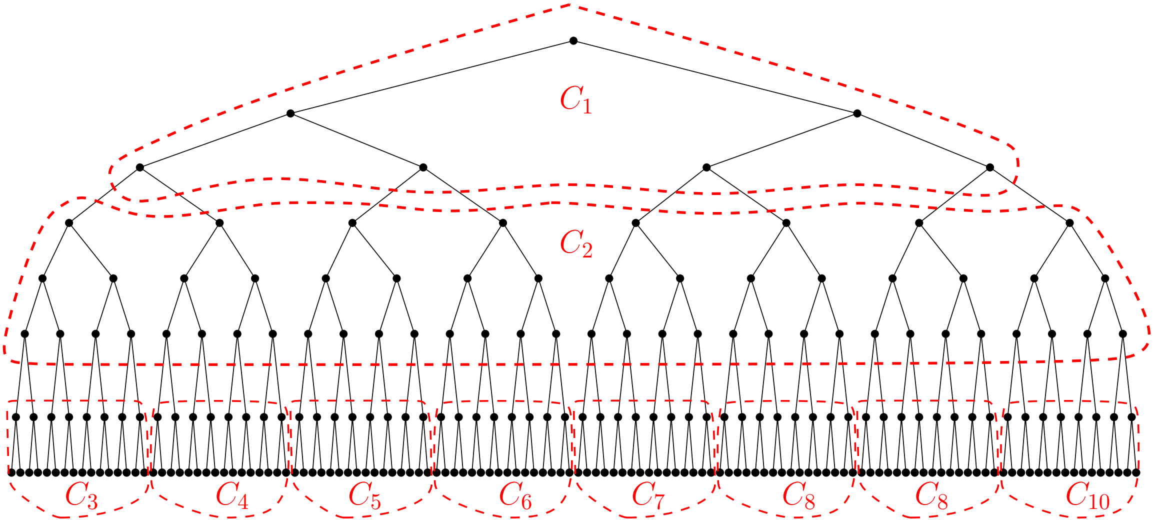
$\forall n > 1$, *there are trees* T_1, T_2 *such that,*

- T_1 *do not admit* $\left(\frac{\log n}{\log\log n}, \log n\right)$*-strong sparse partition scheme.*
- T_2 *do not admit* $\left(\sqrt{\log n}, 2^{\sqrt{\log n}}\right)$*-strong sparse partition scheme.*



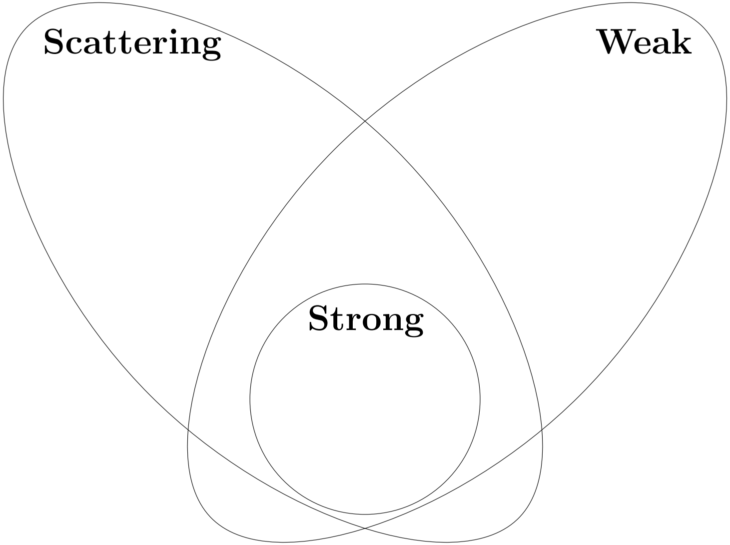
Theorem ([Fil 20])

Every **tree** *is* $(2, 3)$-**scatterable**.



Theorem ([Fil 20])

Every **tree** *admits a* $(4,3)$-**weak** *sparse partition scheme.*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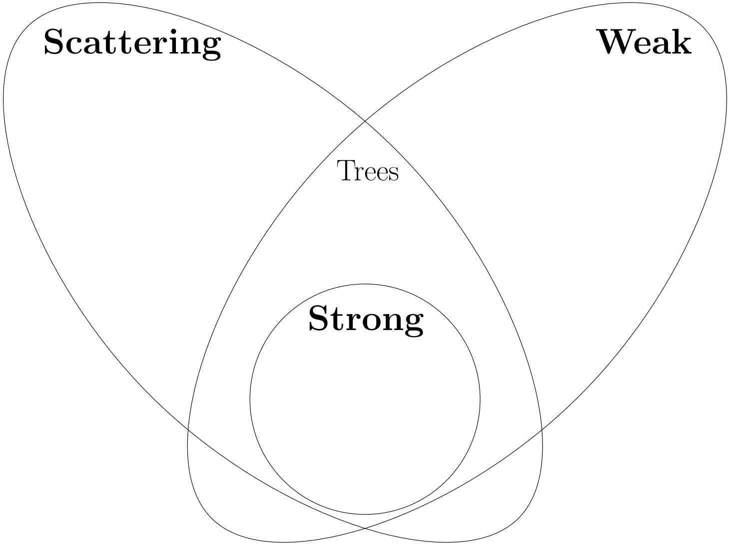
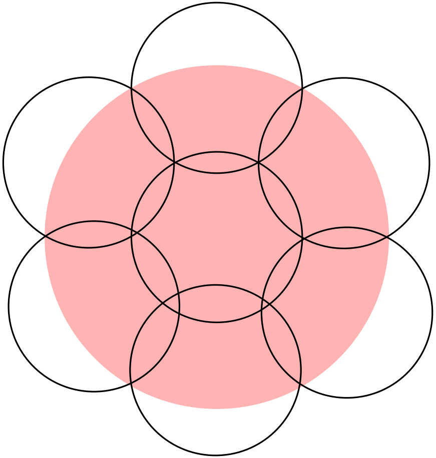


Scattering
Weak
Trees
Strong

Doubling Metrics

Metric space has **doubling dimension $\boldsymbol{d}$** if
every radius r ball can be **covered** by $2^{\boldsymbol{d}}$ balls of radius $\frac{r}{2}$.



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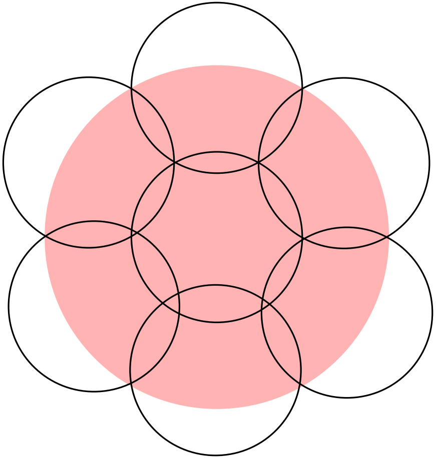

Example: Every d-dimensional Euclidean space has doubling dimension $\boldsymbol{O(d)}$.

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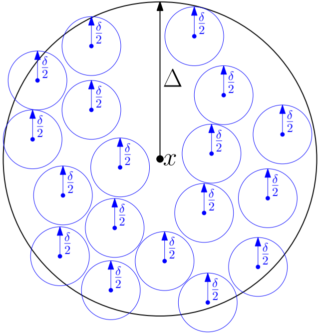

Packing Property

$N \subseteq X$ set s.t. $x, y \in N$ it holds that $d(x,y) \geq \delta$. **Then** $\forall x, R$,

$$|B(x,R) \cap N| \leq \left({}^{R}/_{\delta}\right)^{O(\boldsymbol{d})} .$$

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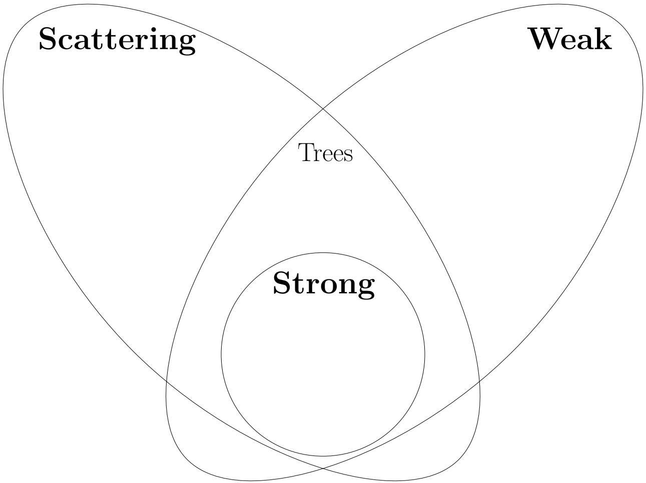
The graph $G = (V, E, w)$ has doubling dimension $O(\boldsymbol{d})$,
if (V, d_G) (the shortest path metric) has doubling dimension $O(\boldsymbol{d})$.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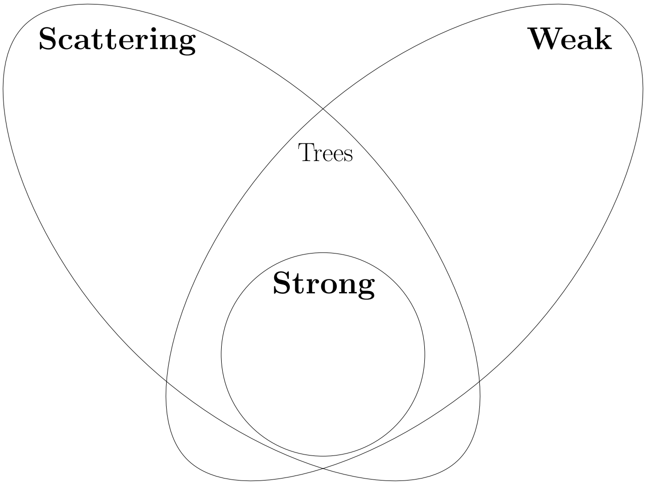
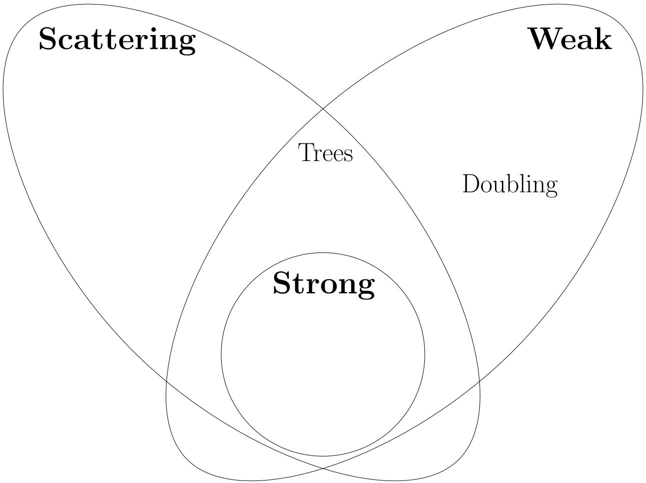


Theorem ([JLNRS 05])

Every graph with **doubling dimension d** *admits a* $(1, 2^{O(d)})$-**weak** *sparse partition scheme.*



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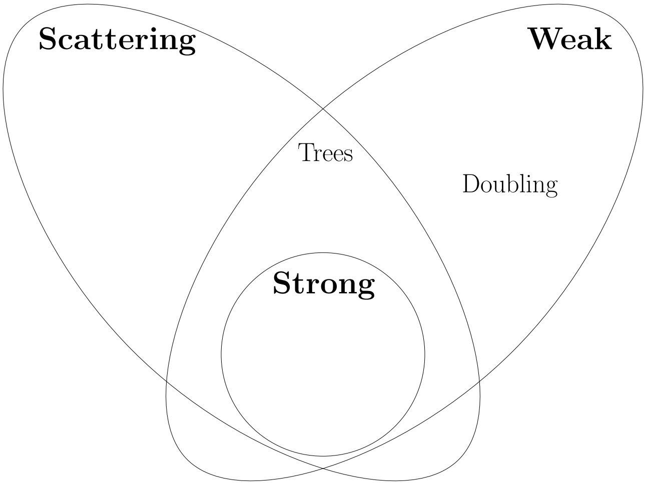
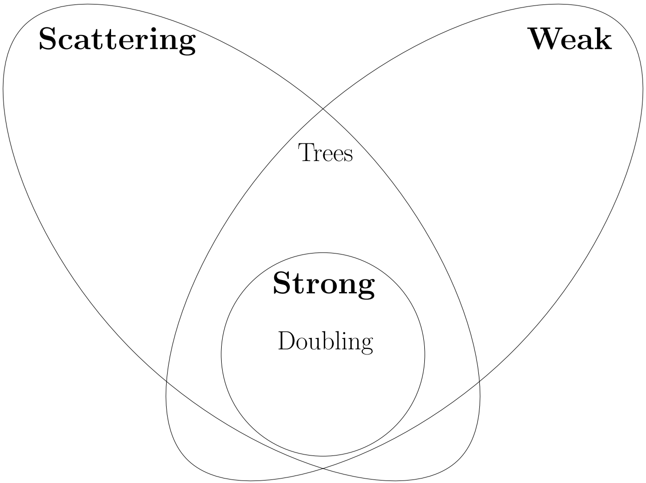


Theorem ([Fil 20])

Every graph with **doubling dimension d** *admits a* $(O(d), \tilde{O}(d))$-**strong** *sparse partition scheme.*



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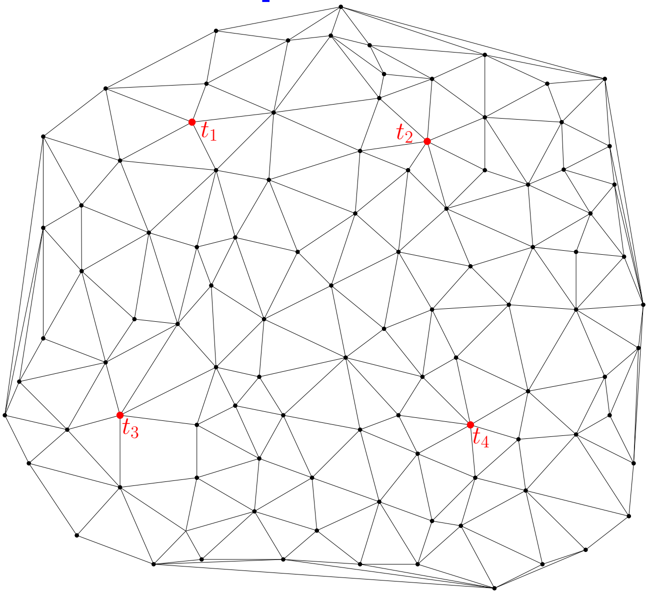
MPX [Miller, Peng, Xu 2013]



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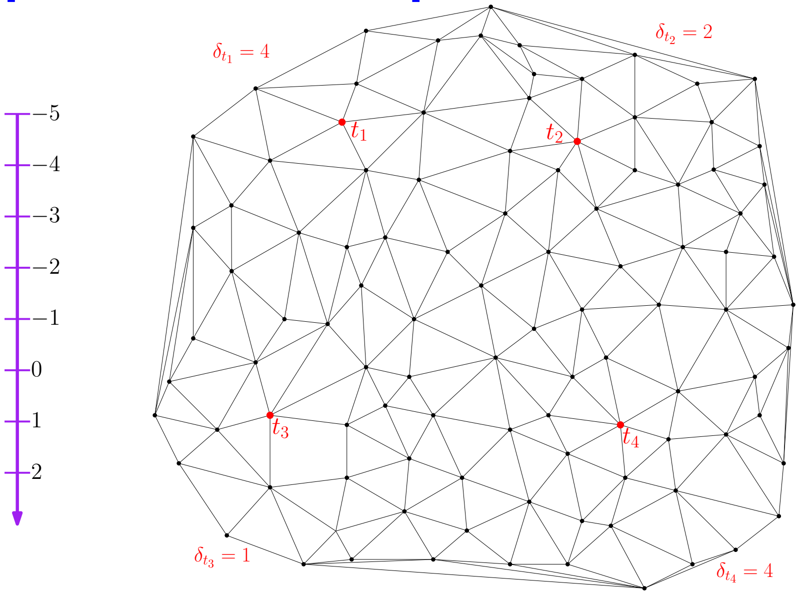

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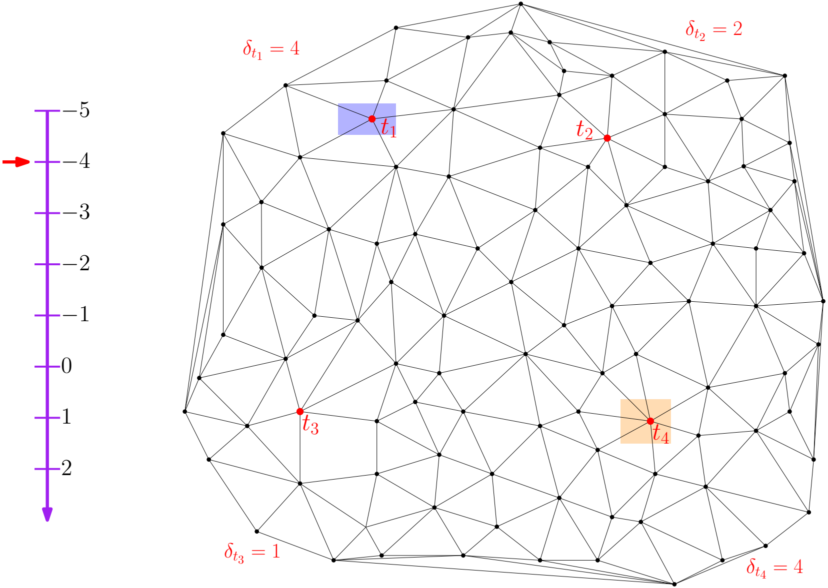

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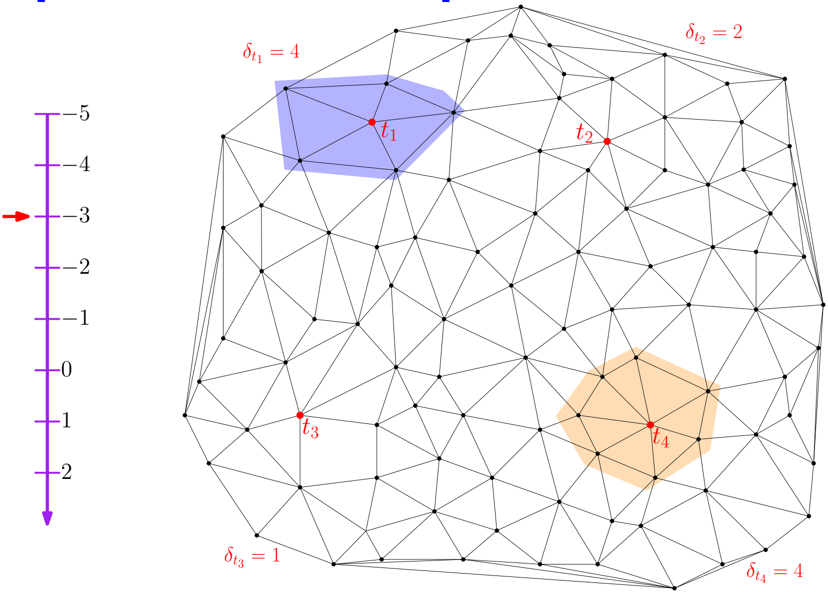

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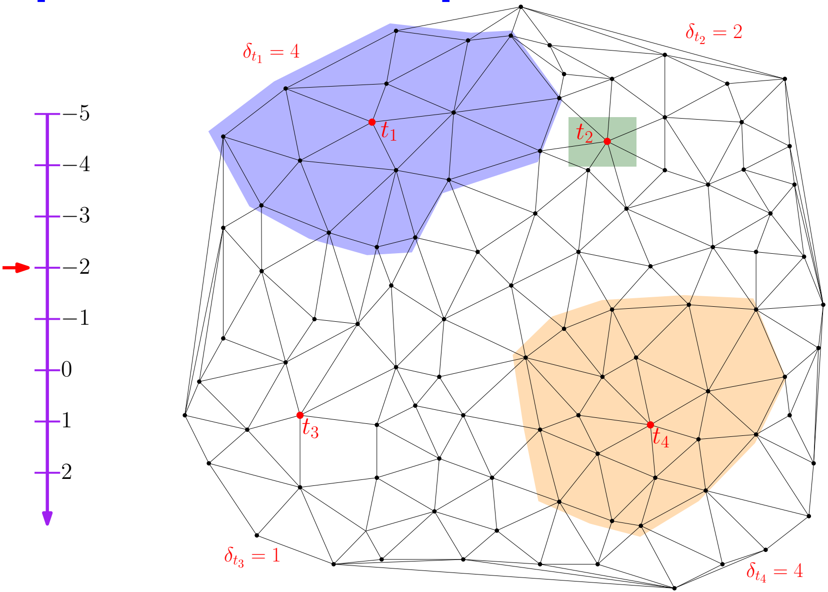

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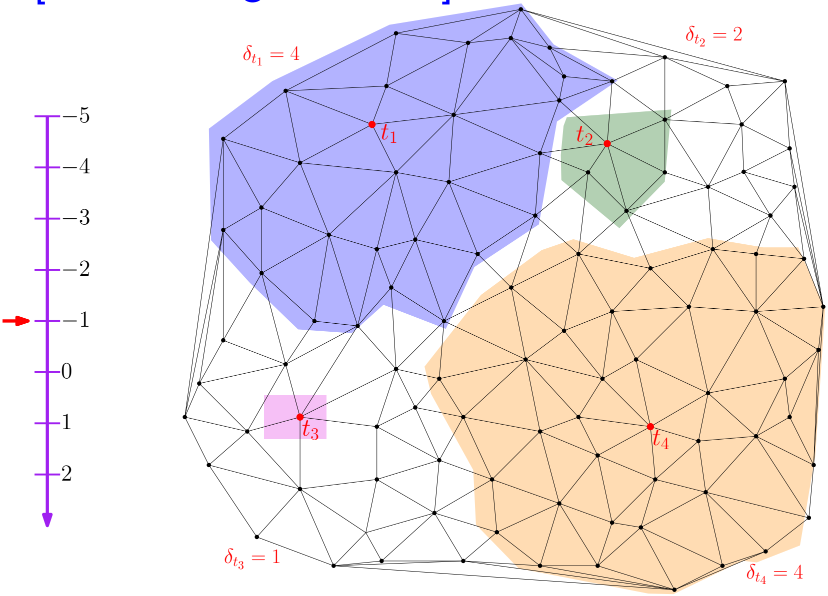

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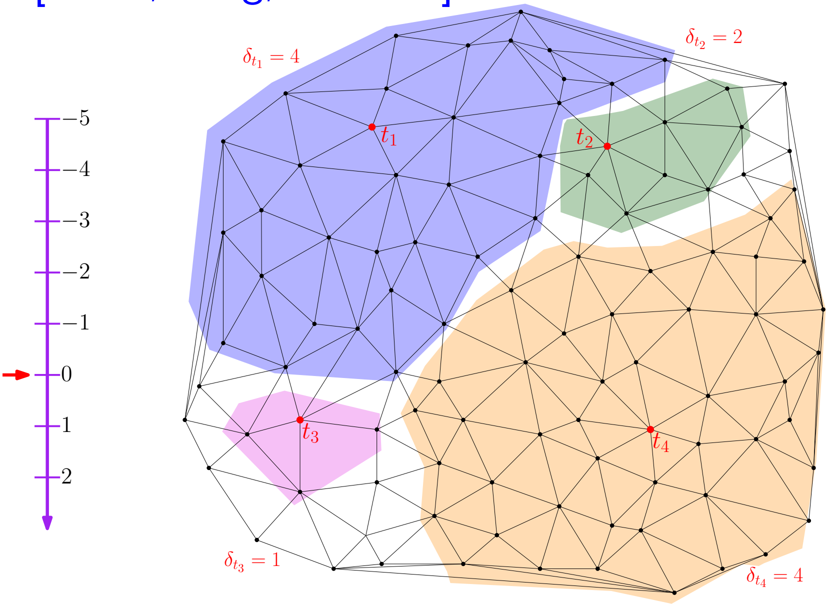

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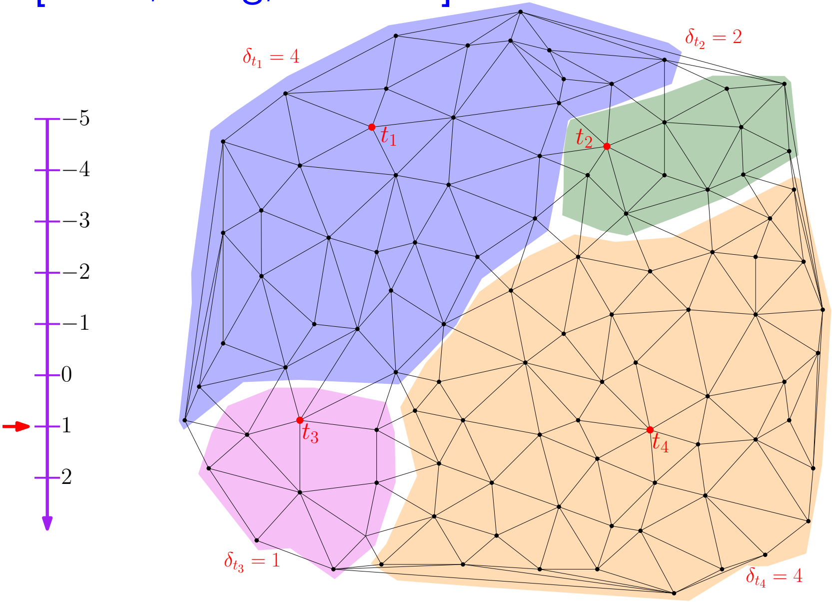

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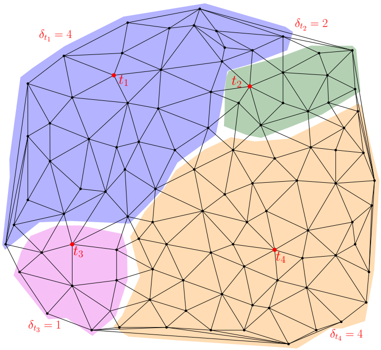

Inherently **connected**!

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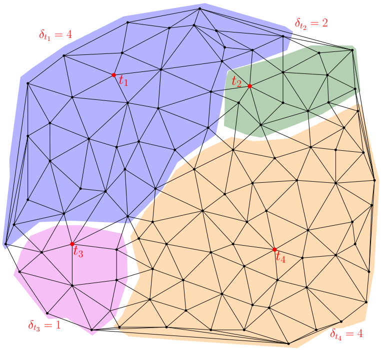

Inherently **connected**!

Formally, for $v \in V$ set $\boldsymbol{f_v}(t) = \delta_t - d_G(v, t)$.

v **joins** the cluster C_t of the center t **maximizing** f_v.

Partition Algorithm

Algorithm: 1. Let N be a Δ-net.

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Definition (Δ-net)

Set N s.t.:

- $\forall u, v \in N$, $d_G(u, v) > \Delta$.
- $\forall v \in V$ there is a net point $u \in N$ s.t. $d_G(u, v) \leq \Delta$.

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$X = \min\{X', \lambda_T\}$ where $X' \sim \mathrm{Exp}(\lambda)$.

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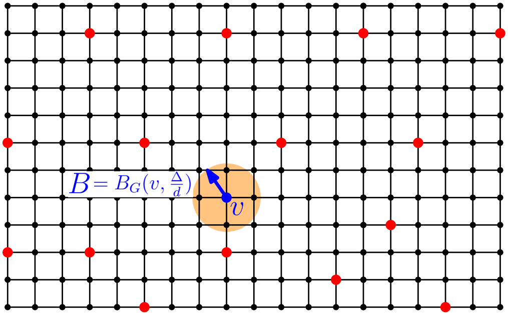

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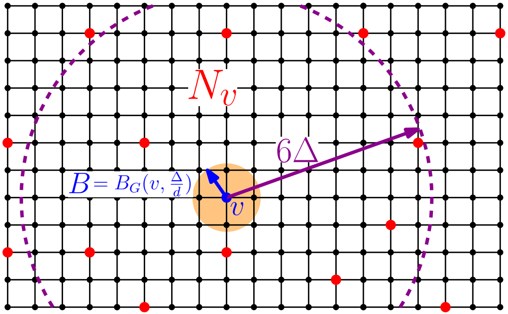

Set $N_v = N \cap B_G(v, 6\Delta)$. By packing argument: $|N_v| = 2^{O(d)}$.

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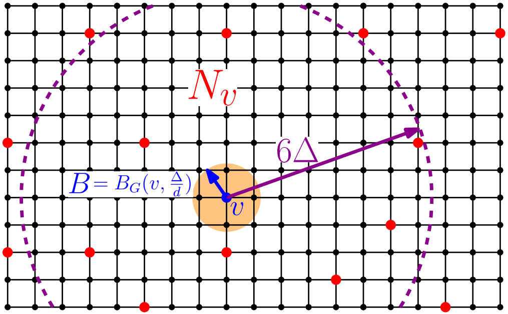

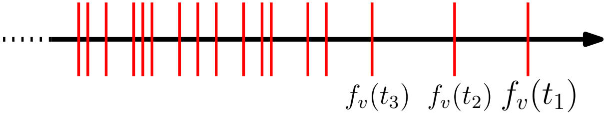

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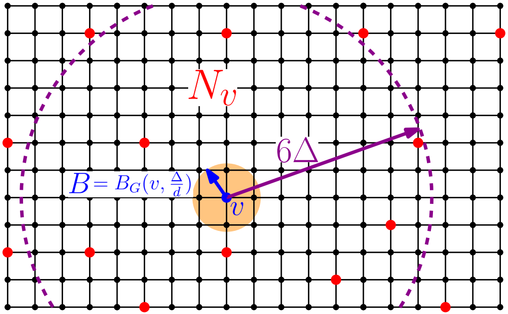

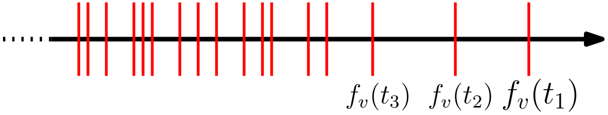

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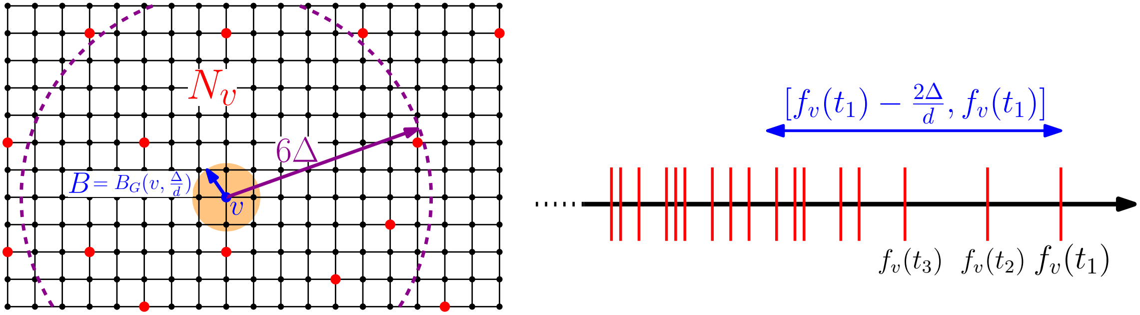

Consider $u \in B$, for all $t \in N$, $\quad |f_v(t) - f_u(t)| \le d_G(u,v) \le \frac{\Delta}{\mathbf{d}}$

B can intersects center t' **only if** $f_v(t') \ge f_v(t_1) - \frac{2\Delta}{\mathbf{d}}$.

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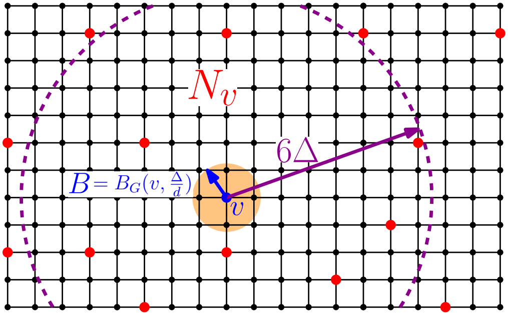
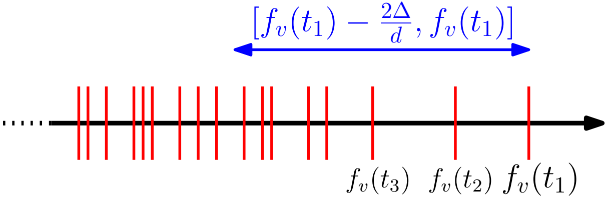
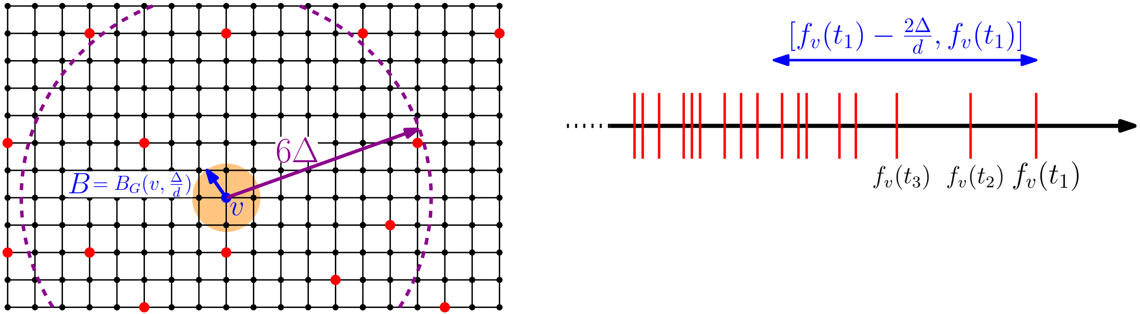




For how many $t \in N_v$, $f_v(t) \in [f_v(t_1) - \frac{2\Delta}{\boldsymbol{d}}, f_v(t_1)]$?



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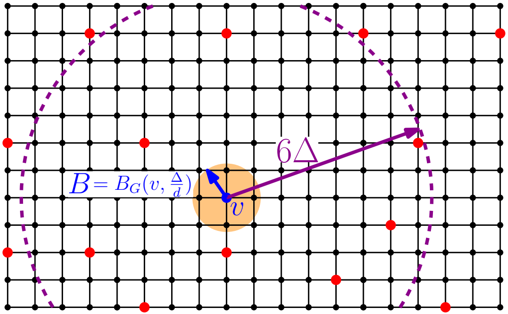

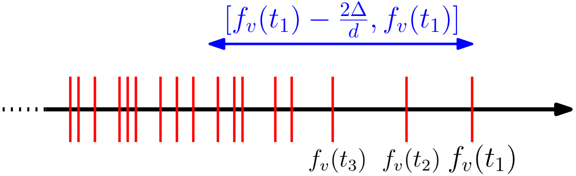

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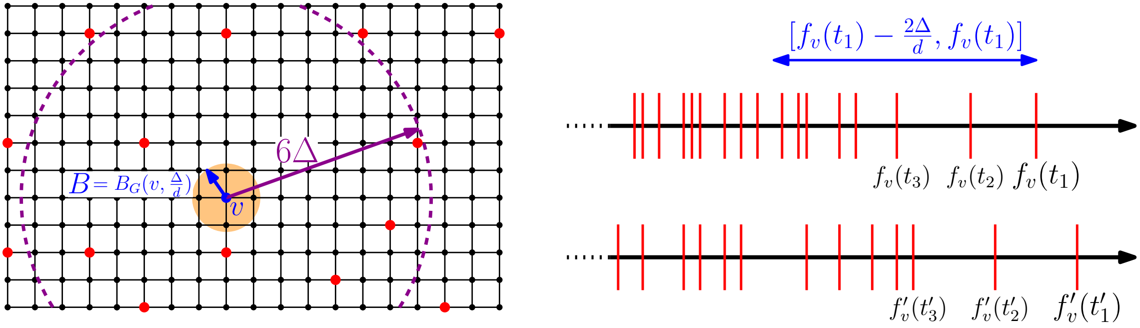

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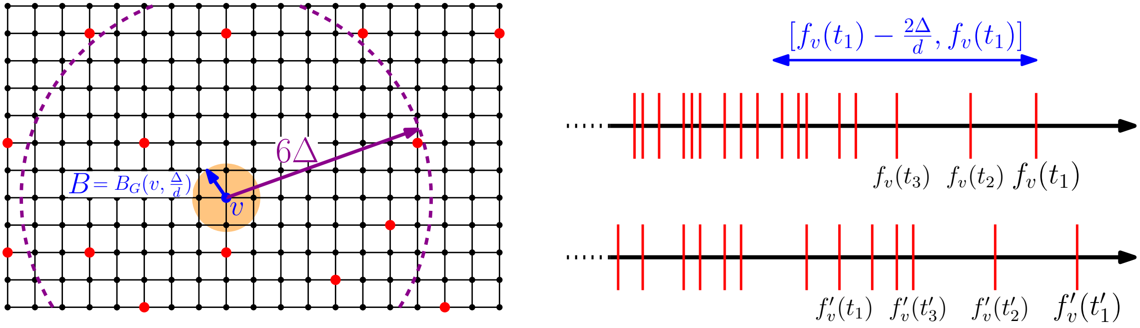

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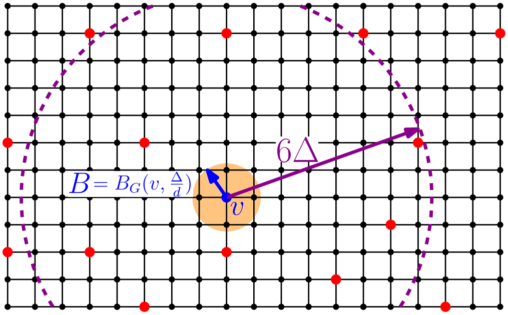

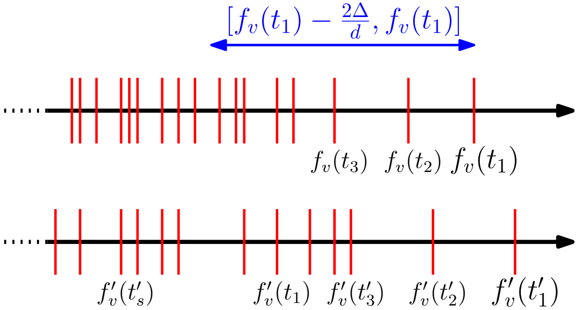

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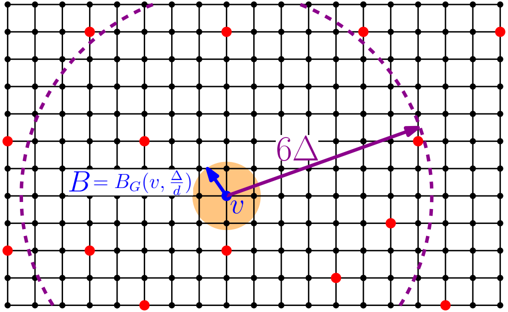

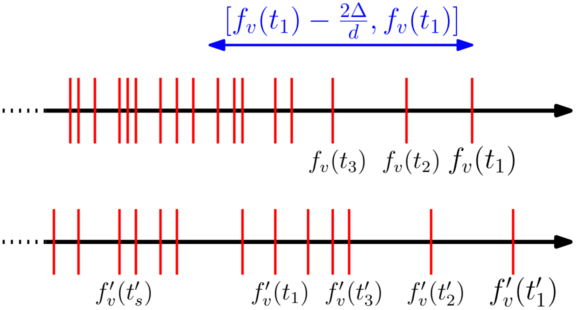

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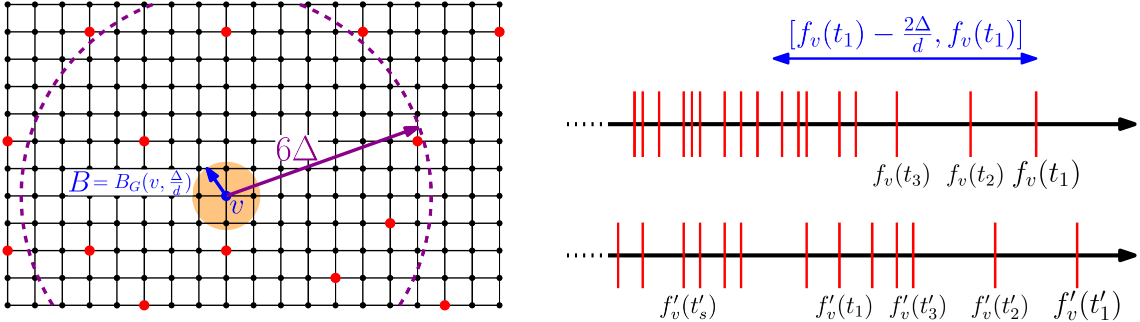

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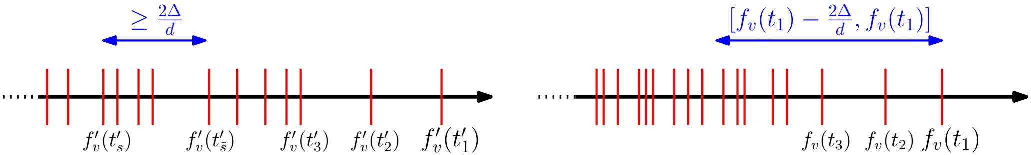

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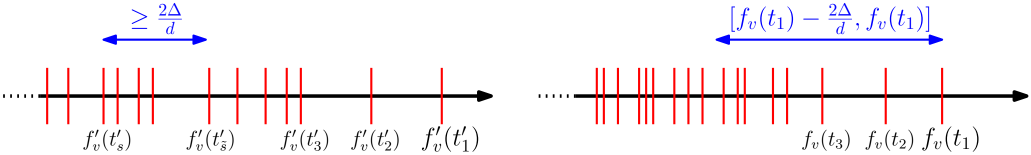

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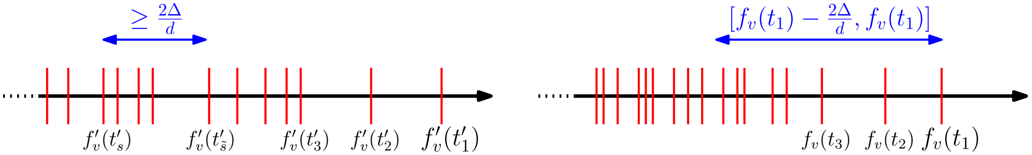
$t \in N_v$ is **betailed** with probability $\Pr[\delta_t = \lambda_T] = \Pr[\delta'_t \geq \lambda_T] = e^{-\frac{\lambda_T}{\lambda}} = e^{-4\boldsymbol{d}}$.



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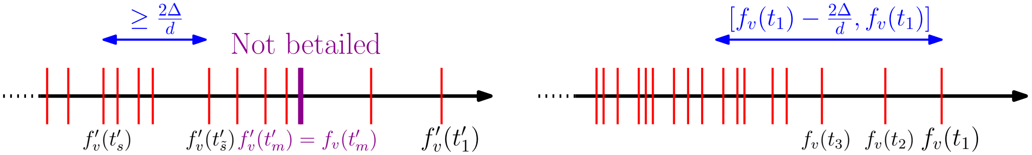
The probability that at least $\tilde{s}$ centers in N_v are betailed bounded by

$$|N_v|^{\tilde{s}} \cdot \left(e^{-4\boldsymbol{d}}\right)^{\tilde{s}} = 2^{O(\boldsymbol{d}\tilde{s})} \cdot \left(e^{-4\boldsymbol{d}\tilde{s}}\right) = e^{-\Omega(\boldsymbol{d}\tilde{s})}$$



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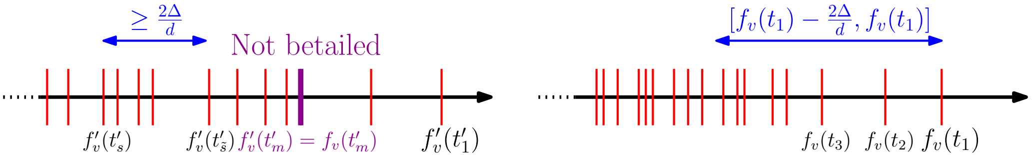

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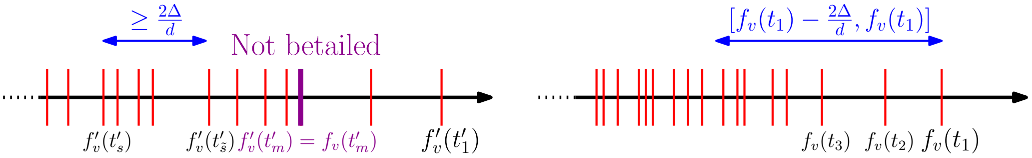

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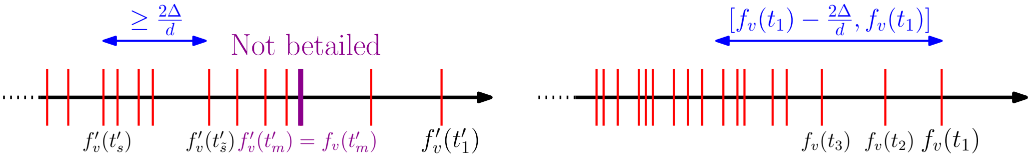

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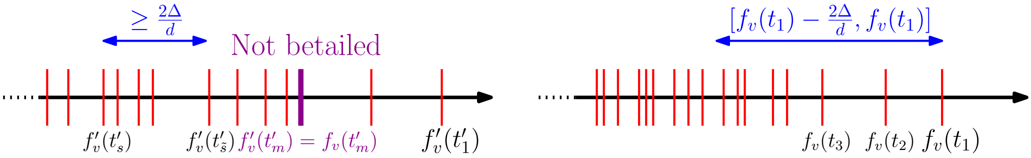

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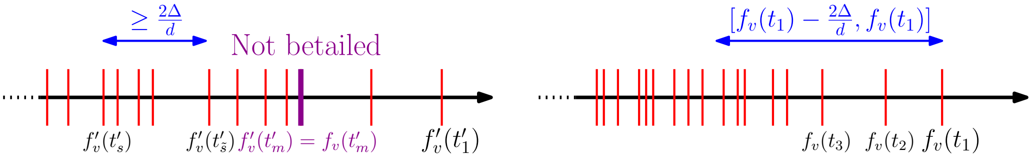

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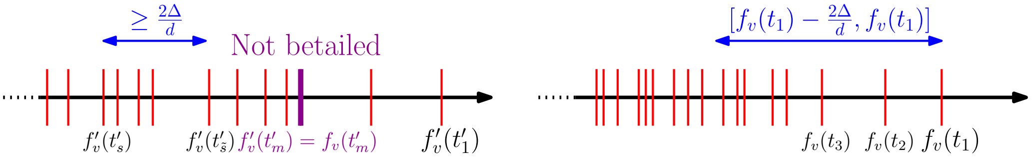

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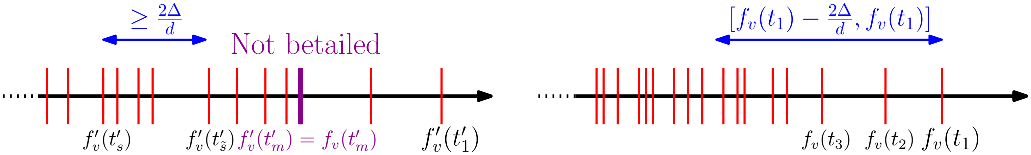

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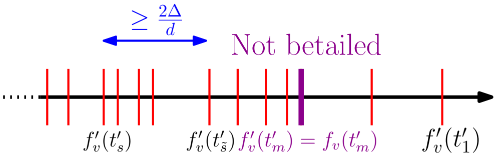
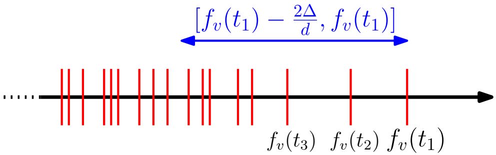
Corollary

W.h.p. $B = B_G(v, \frac{\Delta}{\boldsymbol{d}})$ intersects at most $s = O(\boldsymbol{d})$ clusters.





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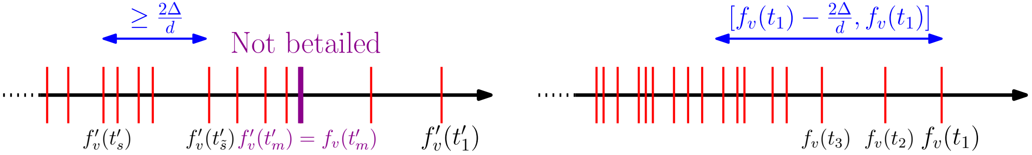
Corollary

W.h.p. $B = B_G(v, \frac{\Delta}{\boldsymbol{d}})$ *intersects at most* $s = O(\boldsymbol{d})$ *clusters.*

Using the **L**ovász **L**ocal **L**emma, we conclude



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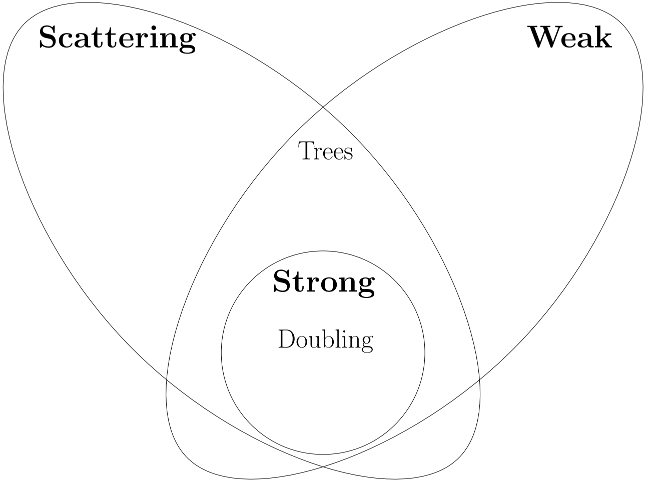
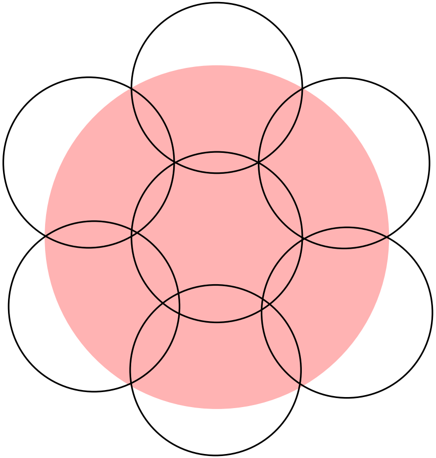
Using the **L**ovász **L**ocal **L**emma, we conclude

Theorem ([**Fil** 20])

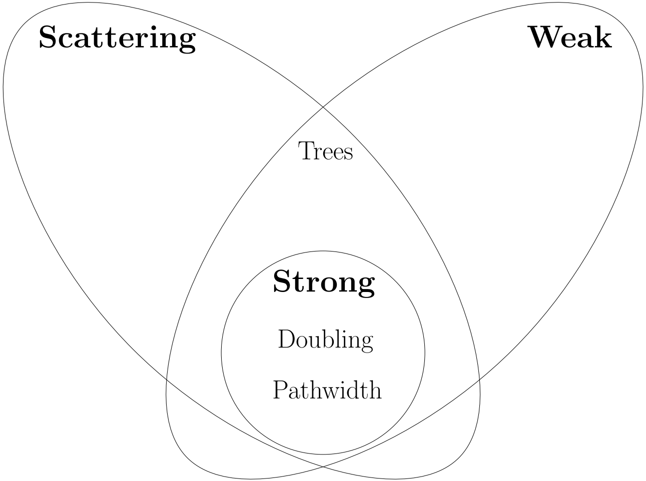
Every graph with **doubling dimension** $\boldsymbol{d}$ *admits a $(O(\boldsymbol{d}), \tilde{O}(\boldsymbol{d}))$-***strong** *sparse partition scheme.*

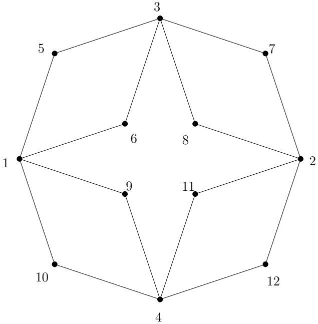


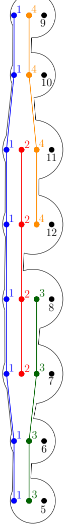


Theorem ([**Fil** 20])

Every graph with **doubling dimension *d*** *admits a* $(O(d), \tilde{O}(d))$-**strong** *sparse partition scheme.*

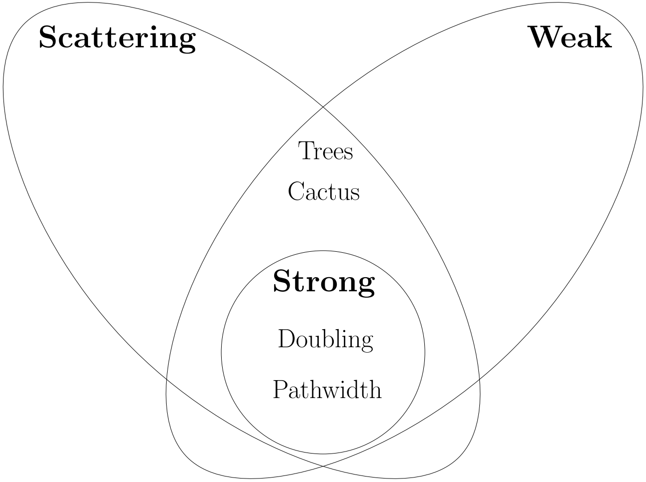


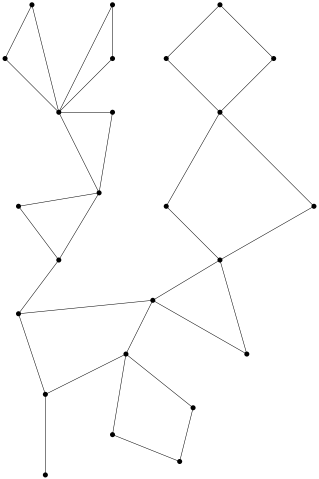




Theorem ([Fil 20])

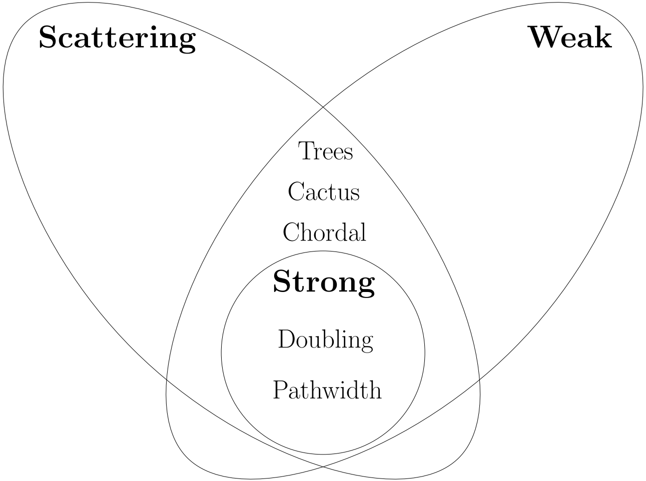
Every graph with **pathwidth** ρ *admits a* $(O(\rho), O(\rho^2))$-**strong** *sparse partition scheme, and a* $(8, 5\rho)$-**weak** *sparse partition scheme.*

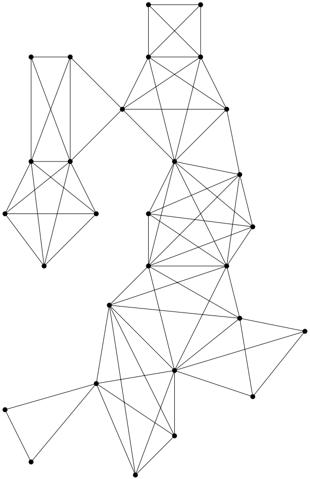




Theorem ([Fil 20])

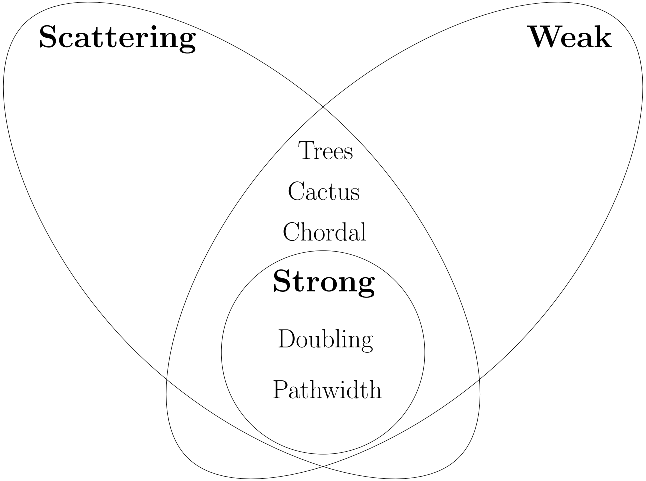
Every **cactus** *graph admits a* $(4,5)$-**scattering** *partition scheme,*
and a $(O(1), O(1))$-**weak** *sparse partition scheme.*

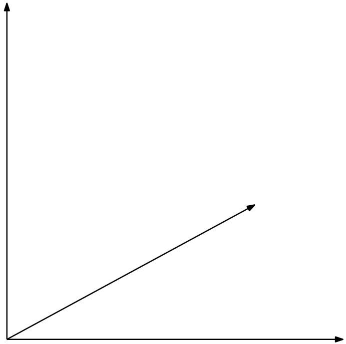




Theorem ([Fil 20])

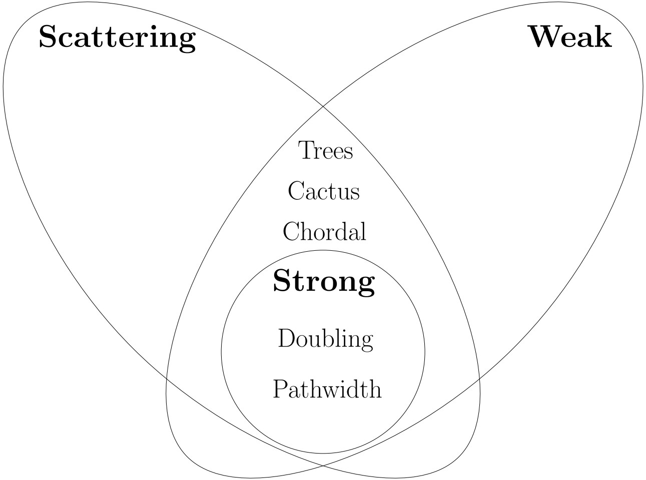
Every **chordal** *graph admits a* (2, 3)-**scattering** *partition scheme, and a* (24, 3)-**weak** *sparse partition scheme.*

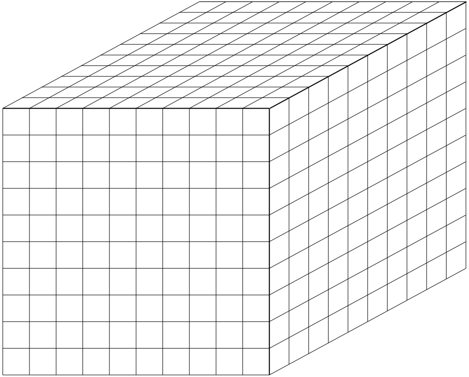




Theorem ([Fil 20])

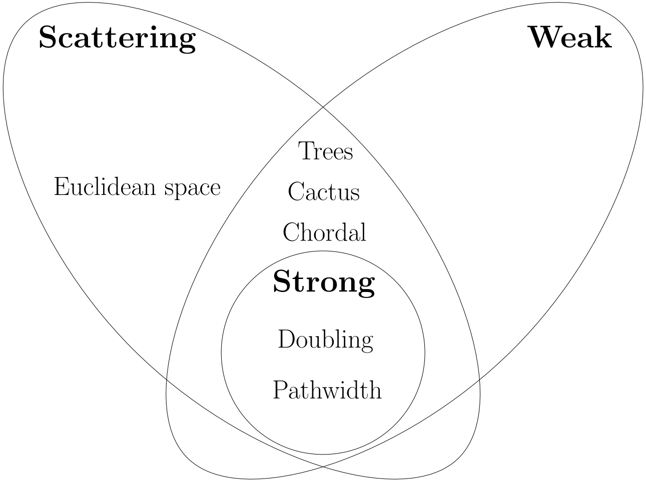
*Suppose that the space $(\mathbb{R}^d, \|\cdot\|_2)$ admits a (σ,τ)-**weak** sparse partition scheme. Then $\boldsymbol{\tau} \geq (1+\frac{1}{2\sigma})^d$ (alternatively $\sigma > \frac{d}{4\ln\tau}$).*

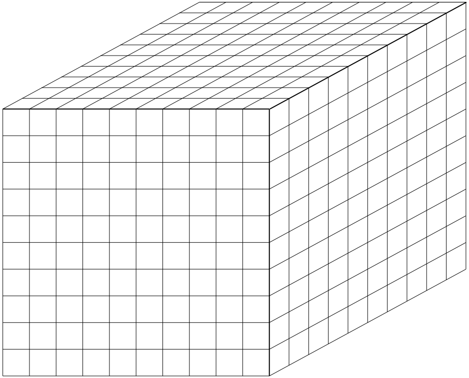




Theorem ([Fil 20])

The space $(\mathbb{R}^d, \|\cdot\|_2)$ *admits a* $(1, 2d)$-**scattering** *partition scheme.*

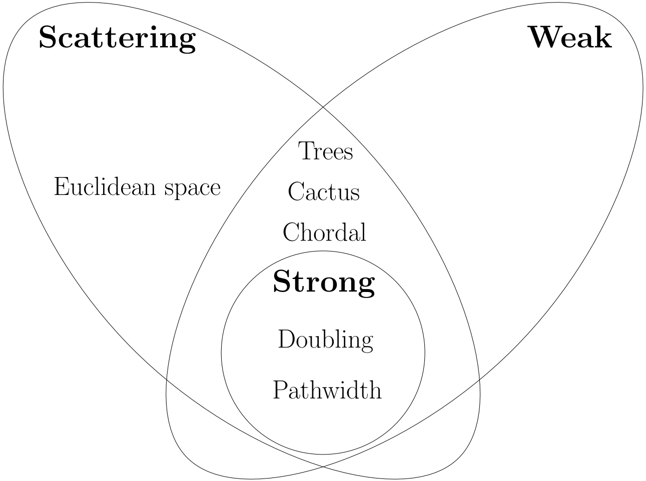


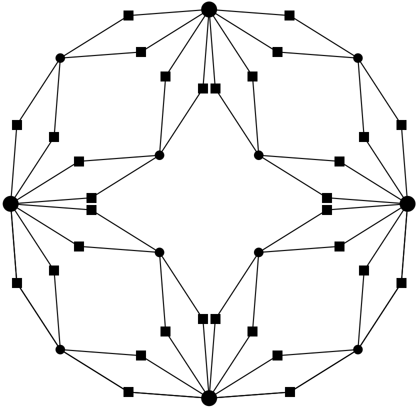


Theorem ([Fil 20])

The space $\left(\mathbb{R}^d, \|\cdot\|_2\right)$ *admits a* $(1, 2d)$-**scattering** *partition scheme.*

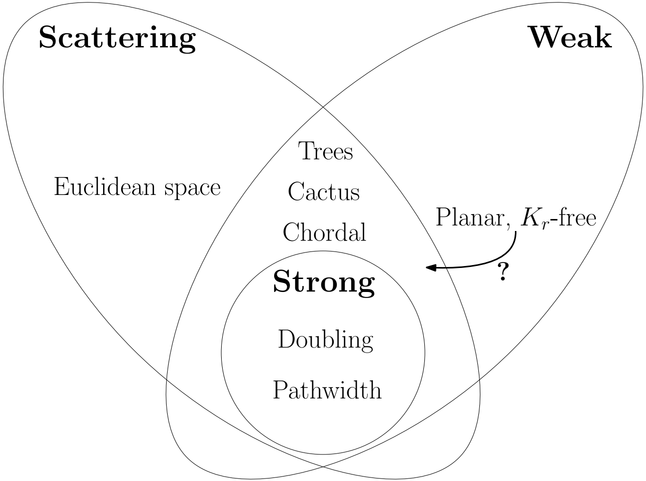
(For weak: $\boldsymbol{\tau} \geq (1 + \frac{1}{2\sigma})^d \Rightarrow$ *no* $(O(1), 2^{\Omega(d)})$*-weak partition scheme).*

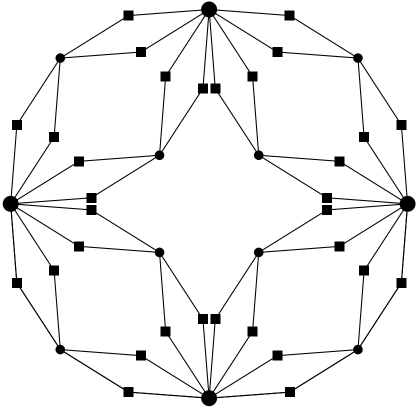




Theorem ([**Fil** 20])

*Every $K_{r,r}$-free graph admits an $(O(r^2), 2^r)$-**weak** sparse partition scheme.*





Theorem ([Fil 20])

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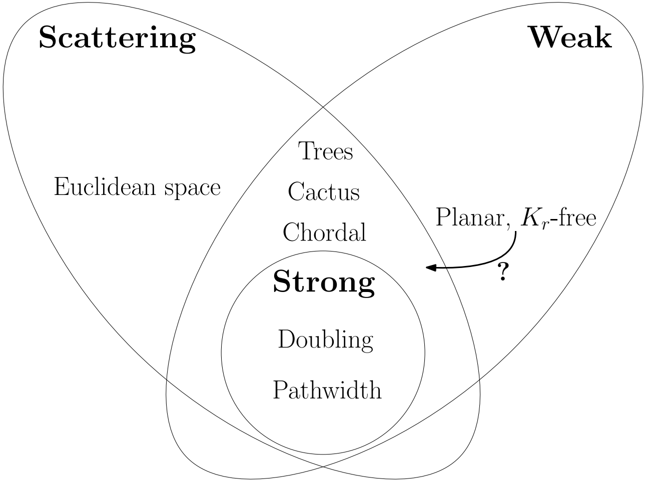
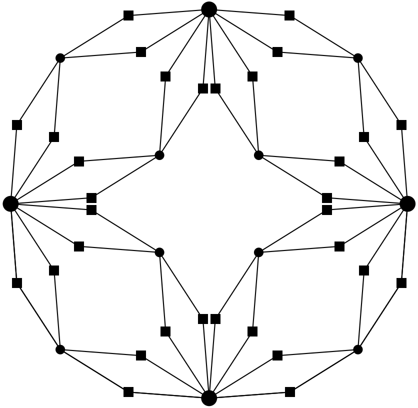
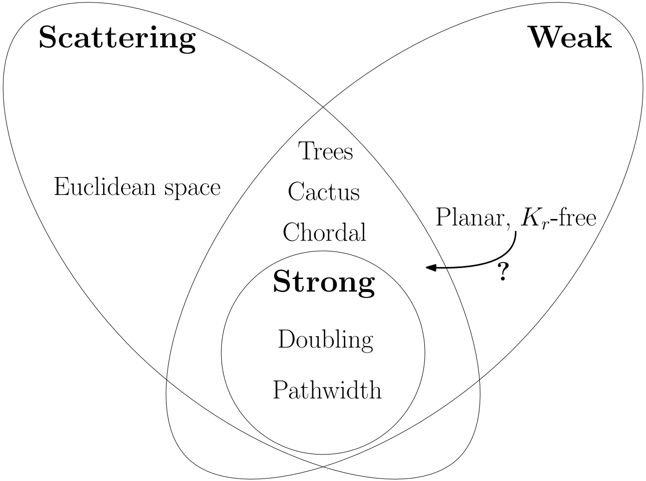
*What about **scattering**?*

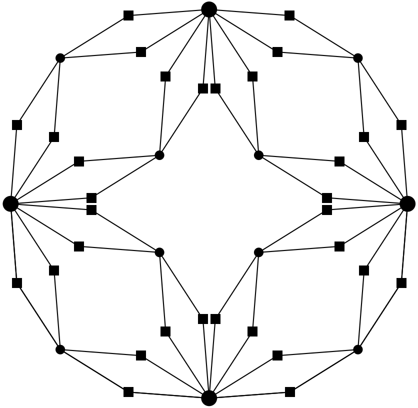




Conjecture

Planar *graphs are* $(O(1), O(1))$-**scattering**.





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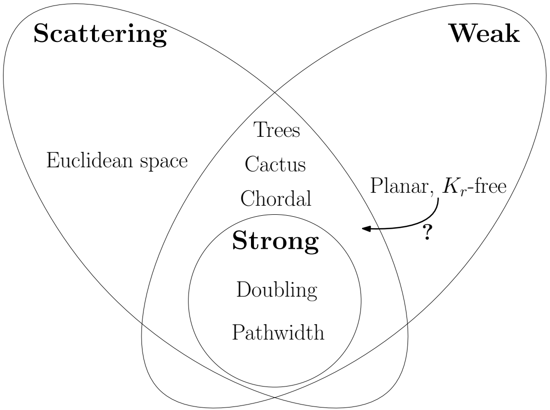
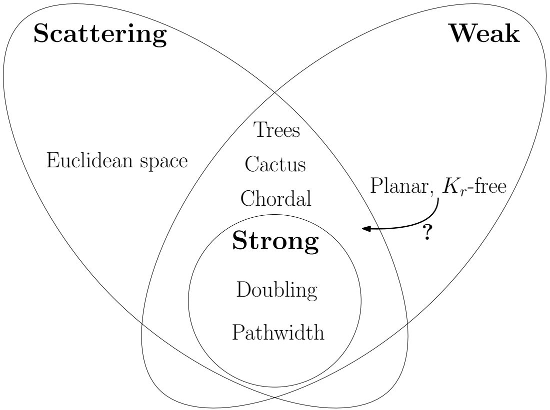
Will imply a solution for the **SPR** problem with **distortion** $O(1)$ for **planar** graphs!



Consider a **general** weighted n vertex graph G:

- [JLNRS 05]: G admits $(O(\log n), O(\log n))$-**weak** sparse partition scheme.



Consider a **general** weighted n vertex graph G:

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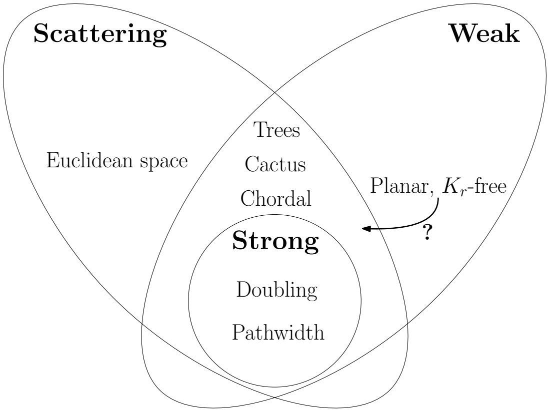
- [KKN 14] (implicitly): G admits $(O(\log n), O(\log n))$-**scattering** partition scheme.



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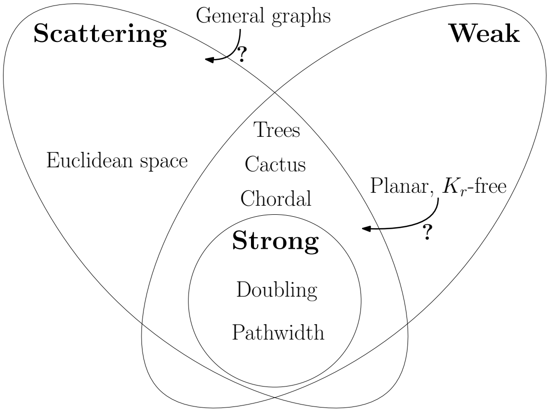

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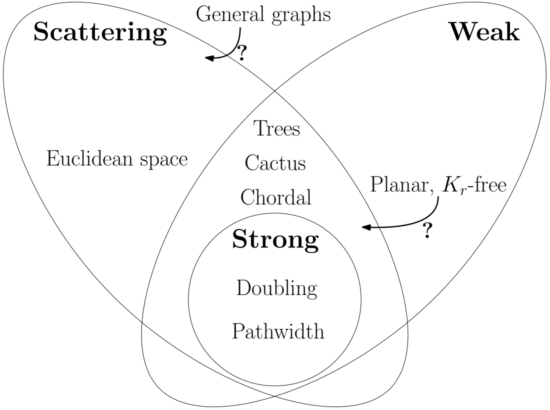
- [**Fil** 20]: G admits $(O(\log n), O(\log n))$-**strong** sparse partition scheme.
- [**Fil** 20]: $\exists G$ which **do not** admit $(O(\frac{\log n}{\log\log n}), O(\log n))$-**weak** sparse partition scheme.



Conjecture

*Every n vertex graph admits $(O(1), O(\log n))$-**scattering** partition scheme. Furthermore, this is tight.*

Theorem ([JLNRS 05])

*Suppose G admits (σ, τ)–**weak sparse** partition scheme,*
*$\Rightarrow$ solution to the **UST** problem with stretch $O(\tau\sigma^2 \log_\tau n)$.*

Theorem ([Fil 20])

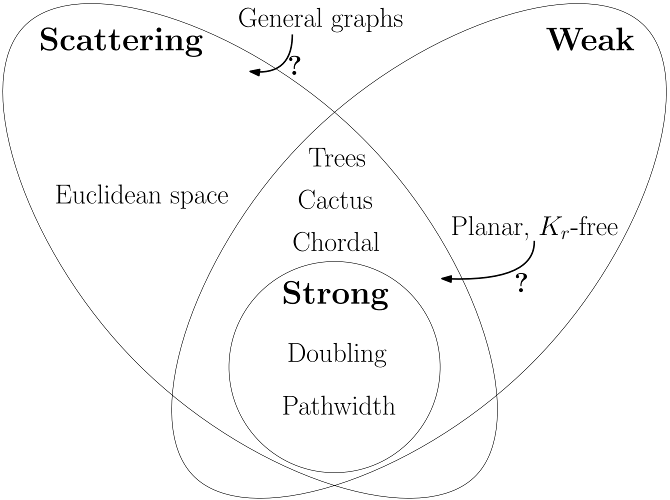
*Suppose that every **induced subgraph** $G[A]$ of G admits (σ, τ)-scattering partition scheme,*
*$\Rightarrow$ solution to the **SPR** problem with distortion $O(\tau^3\sigma^3)$.*

Conjecture

Planar *graphs are* $(O(1), O(1))$-**scattering**.

Conjecture

General n *vertex graph are* $(O(1), O(\log n))$-**scattering**. *Furthermore, this is tight.*



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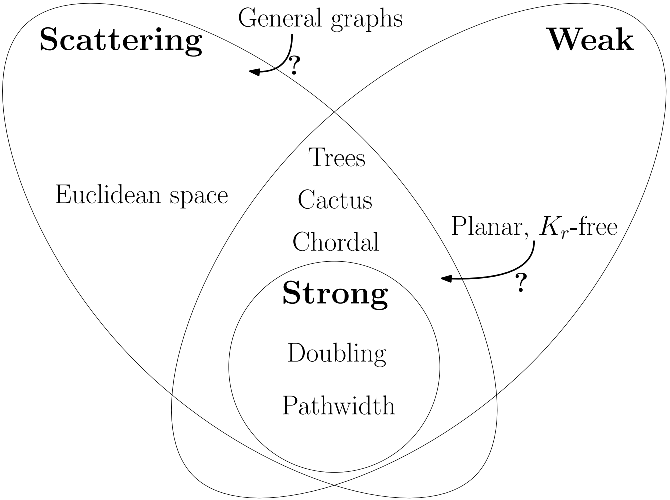

Thank you for listening!