AA Lecture 24   4/8/21

Multiplicative weights Update

n experts.

$t = 1 .. T$:

$$f_i^t = \begin{cases} 1, & \text{if expert } i \text{ is wrong @ } t \\ 0, & \text{oth.} \end{cases}$$

m_i^t = # errs by exp i until t.

M^t = # our errs.

Goal: bound M^T vs best expert

$$\underset{i \in [n]}{\operatorname{argmin}} \; m_i^T .$$

ideal: $M^T \le$.

Examples bad algos

(1) Majority: $n = 3$

B	B	B	--	$m_1 = T$
B	B	B	--	$m_2 = T$
S	S	S		$m_3 = 0$
True: S	S	S		

(2) Follow yesterday:

$n=2$:

1	B	S	B	S	...
2	S	B	S	B	...
True:	B	B	B	B	...

Weighted Majority Algo.

$w_i^1 = 1$ weight of exp. i @ $t=1$

for $i = 1..T$:

$$\sigma_B^t = \sum_{\substack{i:\ \text{exp. } i \text{ says} \\ \text{"buy" on day } t}} w_i^t$$

$$\sigma_S^t = \sum_{i:\ \text{exp. } i \text{ says "S"}} w_i^t$$

if $\sigma_B^t > \sigma_S^t$: "buy @ t"

else: "sell @ t".

for $i = 1..n$:

$f_i^t = 1$ if exp. i is wrong.

$$w_i^{t+1} = w_i^t \cdot (1 - \varepsilon \cdot f_i^t)$$

ε = param.

<u>Th</u>: $M^T \le 2\cdot(1+\varepsilon)\cdot \underline{m_i^t} + \dfrac{2\ln n}{\varepsilon}, \quad \forall i \in [n].$

$\Rightarrow \quad \min_i m_i^t.$

<u>Pf</u>: $\varphi^t = \sum_{i=1}^{n} w_i^t.$

$\varphi^1 = n.$

Fix $i \in [n]$: $\quad \ldots m_i^T \ldots \le \varphi^{T+1} \le \ldots M^T \ldots$

① $\varphi^{T+1} \ge w_i^{T+1} = 1\cdot(1-\varepsilon)^{m_i^T}.$

② $\varphi^{T+1} \le n\cdot\left(1-\frac{\varepsilon}{2}\right)^{M^T}.$

$\varphi^1 = n.$

suppose we make error @ time t.

$\Rightarrow \sigma_{corr} \le \sigma_{wrong}$

$$\sum_{i:\text{ correct exp}} w_i^t \le \sum_{i:\text{ wrong exp.}} w_i^t$$

$$\varphi^{t+1} = \sum_{i:\text{ correct exp}} w_i^t + \sum_{i:\text{ wrong exp.}} w_i^t (1-\varepsilon)$$

$$\leq \varphi^t - \varepsilon \cdot \sum_{i:\text{ wrong exp}} w_i^t$$

$$\leq \varphi^t - \frac{\varepsilon}{2} \cdot \varphi^t$$

$$= \varphi^t (1 - \varepsilon/2).$$

1 + 2:

$$(1-\varepsilon)^{m_i^T} \leq \varphi^{T+1} \leq n \cdot \left(1 - \frac{\varepsilon}{2}\right)^{M^T}$$

$$n \cdot \left(1 - \frac{\varepsilon}{2}\right)^{M^T} \leq n \cdot e^{-\frac{\varepsilon}{2} M^T}.$$

$$\ln(1-\varepsilon) \geq -\varepsilon - \varepsilon^2.$$

$$m_i^T \cdot (-\varepsilon - \varepsilon^2) \leq \ln n - \frac{\varepsilon}{2} M^T$$

$$\frac{\varepsilon}{2} M^T \leq \varepsilon(1+\varepsilon) \cdot m_i^T + \ln n$$

$$M^T \leq 2 \cdot (1+\varepsilon) \cdot m_i^T + \frac{2 \ln n}{\varepsilon}. \quad \blacksquare$$

MWU:

$f_i^t \in [-1, +1]$. $\quad m_i^t = \sum_{j \le t} f_i^t$.

MWU: same as before:

* choose follow expert i:
  i chosen randomly proport. to weight.

$$p_i^t = \frac{w_i^t}{\Phi^t} \qquad p^t = (p_1^t, p_2^t, \dots p_n^t)$$

* $$w_i^{t+1} = w_i^t (1 - \varepsilon f_i^t).$$

Th: $M^T \triangleq \mathbb{E}$ # errors we make.

$$= \sum_{t=1}^{T} \mathbb{E}_i [f_i^t] = \sum_t \sum_i p_i^t \cdot f_i^t = \sum_t \langle p^t, f^t \rangle$$

$$M^T \le m_i^t + \varepsilon T + \frac{\ln n}{\varepsilon}.$$

$p_i^t = \frac{w_i^t}{\Phi^t}$.

Pf: $\Phi^t = \sum_i w_i^t$.

① $\Phi^{T+1} \geq w_i^{T+1} = \prod_{t=1}^{T} (1 - \varepsilon f_i^t)$

$$\cancel{\le \prod_t e^{-\varepsilon f_i^t} = e^{-\varepsilon \sum_t f_i^t} = e^{-\varepsilon m_i^T}}.$$

② $\varphi^1 = n$

$$\varphi^{t+1} = \sum_{i=1}^{n} w_i^t \left(1-\varepsilon f_i^t\right)$$

$$= \sum_{i=1}^{n} w_i^t - \varepsilon \sum_{i=1}^{n} \underbrace{\frac{w_i^t}{\varphi^t}}_{p_i^t} \cdot f_i^t \cdot \varphi^t$$

$$= \varphi^t - \varepsilon \cdot \varphi^t \cdot \langle p^t, f^t \rangle$$

$$= \varphi^t \left(1 - \varepsilon \cdot \langle p^t, f^t \rangle\right)$$

$$\Rightarrow \varphi^{T+1} = \varphi^1 \cdot \prod_{t=1}^{T} \left(1 - \varepsilon \cdot \langle p^t, f^t \rangle\right)$$

$$\varphi^{T+1} \le n \cdot \prod_{t=1}^{T} e^{-\varepsilon \langle p^t, f^t \rangle}$$

$$= n \cdot e^{-\varepsilon \cdot \sum_t \langle p^t, f^t \rangle}$$

$$= n \cdot e^{-\varepsilon M^T}.$$

① + ②: $\prod_t (1-\varepsilon f_i^t) \leq n \cdot e^{-\varepsilon M^T}$.

$$\underbrace{\sum_t \lg(1-\varepsilon f_i^t)}\leq \ln n - \varepsilon \cdot M^T$$

$$\geq \sum_t -\varepsilon f_i^t - \varepsilon^2 (f_i^t)^2 \geq \sum_t -\varepsilon f_i^t - \varepsilon^2$$

$$\geq -\varepsilon \cdot m_i^T - \varepsilon^2 \cdot T.$$

$$\Rightarrow -\varepsilon m_i^T - \varepsilon^2 \cdot T \leq \ln n - \varepsilon \cdot M^T$$

$$\Rightarrow M^T \leq m_i^T + \varepsilon T + \frac{\ln n}{\varepsilon}. \quad \boxtimes$$

for $\forall i \in [n] \Rightarrow M^T \leq \min_i m_i^T + \varepsilon T + \frac{\ln n}{\varepsilon}$.

Remarks if $f_i^t \geq 0 \Rightarrow$

$$M^T \leq (1+\varepsilon) m_i^T + \frac{\ln n}{\varepsilon}.$$

Bandit problems. explore - exploit.