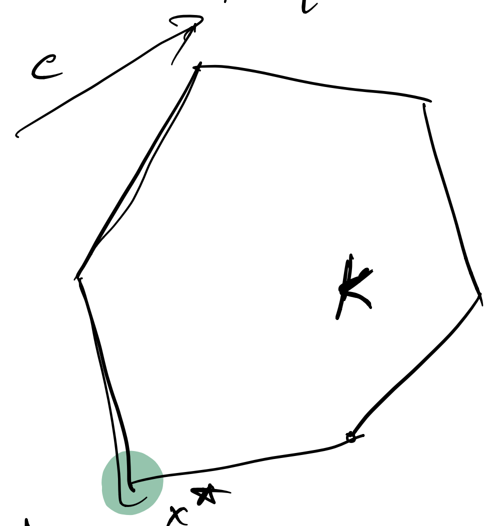
AA Lecture 23 4/6/21

Interior Point Method for LP

LP: $\boxed{\begin{array}{l} \min\ c^T x \\ \text{s.t. } Ax \le b. \end{array}}$

$K = \{x : Ax \le b\}$.



$$f(x) = \begin{cases} c^T x, & x \in K \\ +\infty, & x \notin K \end{cases}$$

Boundary : ∂K.

Define $F(x) = \begin{cases} < \infty, x \in K \\ +\infty, & x \notin K. \end{cases}$ barrier function.

$F(x) \to \infty$ as $x \to \partial K$.

$$f_\eta(x) = \eta \cdot c^T x + F(x), \quad \eta \ge 0.$$

New goal : $\min_{x \in \mathbb{R}^n} f_\eta(x)$.

A_i = i^{th} row of A

$$F(x) = \lg \prod_{i=1}^{m} \frac{1}{b_i - A_i x}$$

LP: K:

$A_i x \le b_i$

$$= \sum_{i=1}^{m} \lg \frac{1}{b_i - A_i x} = \sum_{i=1}^{m} -\lg(b_i - A_i x).$$

<u>Def:</u> $x^*_\eta = \arg\min_x f_\eta(x) = \arg\min \eta c^T x + F(x).$

<u>Claim:</u> $f_\eta(x)$ is convex

~~pf~~: f_η convex $\Leftrightarrow$ $\nabla^2 f_\eta$ pos. semi. def

($\Leftrightarrow$ all $\lambda \geq 0$).

$$\nabla f_\eta(x) = \eta \cdot c^T + \sum_{i=1}^{m} \frac{A_i}{b_i - A_i x}$$

$$\nabla^2 f_\eta(x) = \sum \frac{A_i^T A_i}{(b_i - A_i x)^2}$$

Consider ~~∀~~ y:

$$y^T \cdot \nabla^2 f_\eta(x) \cdot y =$$

$$= \sum \frac{y^T A_i^T \cdot A_i y}{(b_i - A_i x)^2}$$

$$= \sum \frac{\|A_i y\|_2^2}{\underbrace{(b_i - A_i x)^2}_{\text{slack variable } \xi_i \geq 0.}} \geq 0.$$

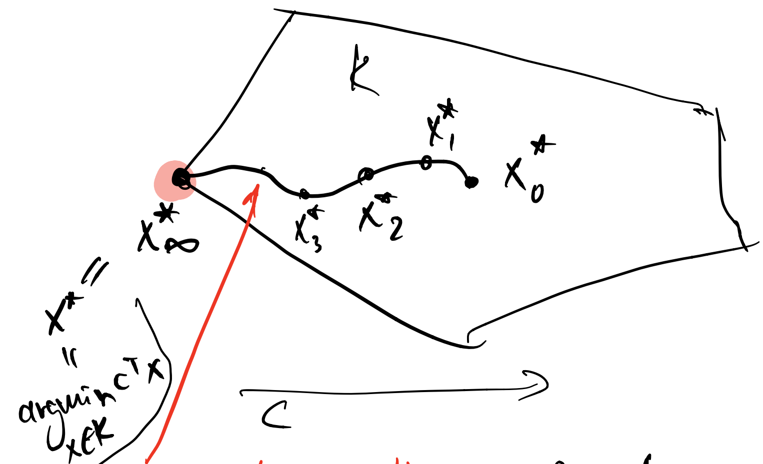
$\boxtimes$

<u>Remarks</u>: if A is fullrank $(\mathrm{vol}(K) > 0)$

$\Rightarrow \lambda_{\min}(\nabla^2 f_\eta(x)) > 0.$

$\Rightarrow f$ is strongly convex

$x^*_\eta = \mathrm{opt}\ f_\eta. \quad \eta \geq 0. \quad f_\eta(x) = \eta c^T x + F(x).$



x^*_0 = analytic center of K.

x^*_η — continuous as func. of η.

central path. $= \{x^*_\eta : \eta \geq 0\}.$

<u>Alg 0</u>: solve x^*_η for η = very large.

<u>GD</u>: depends on cond. # of F. could be too large.

<u>Newton</u>: requires <u>warm start</u>.

<u>Alg 1</u>: walk the central path $\eta \nearrow$.

from $\eta \approx 0$ to η = very large.

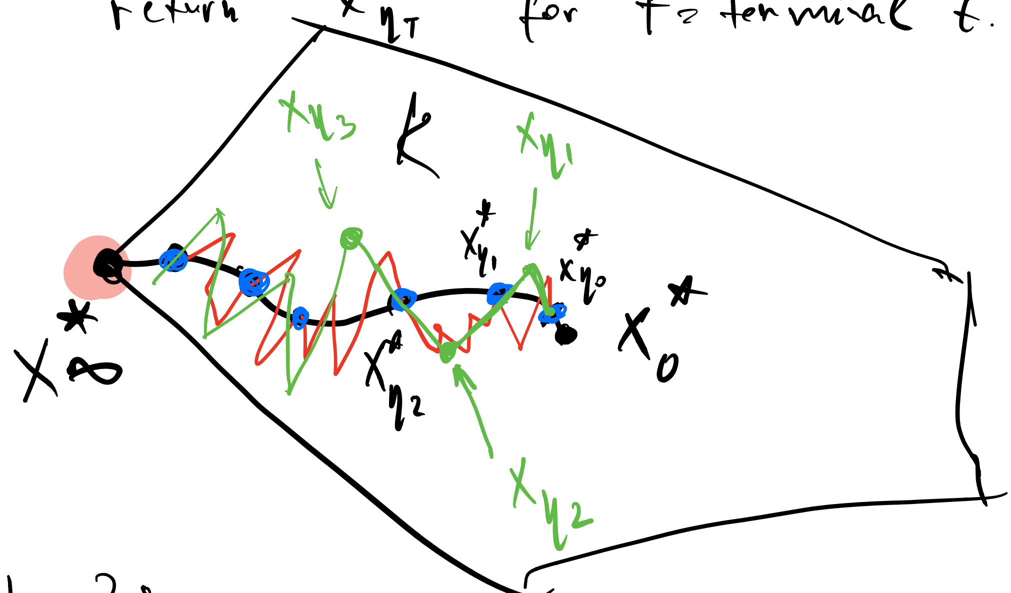
1. Start at $x^*_{\eta_0}$ for $\eta_0 > 0$ small. $[x^*_\eta \approx x^*_0]$.
(assume know x^*_0)

2. For each iteration t:

$$\eta_{t+1} = \eta_t \cdot (1+\alpha), \quad \alpha > 0 \text{ small.}$$

Compute $x^*_{\eta_{t+1}}$ using Newton's method with warm start $= x^*_{\eta_t}$.

3. Terminate once η is "large enough". return $x^*_{\eta_T}$ for T = terminal t.



Alg 2:

2'. for each iter. t:

$$\eta_{t+1} = \eta_t \cdot (1+\alpha)$$

$$x_{\eta_{t+1}} = x_{\eta_t} + \underbrace{n(x_{\eta_t})}_{\text{Newton step.}}$$

Stopping condition: $T = ?$

Claim: $c^T x^*_\eta - c^T x^* \le m/\eta \quad \forall \eta > 0.$

Set $\frac{m}{\eta_T} = \varepsilon \Rightarrow \eta_T = m/\varepsilon.$

$$(1+\alpha)^T \eta_0 = m/\varepsilon$$

$$\Rightarrow T = \theta\left(\lg_{1+\alpha} m/\varepsilon\eta_0\right) = O\left(\frac{\lg m/\varepsilon\eta_0}{\alpha}\right).$$

$$\alpha = \frac{1}{\text{Poly}(n, m)}.$$

Proof: by def x^*_η: $\nabla f_\eta(x^*_\eta) = 0$

$$\Leftrightarrow \eta c + \nabla F(x^*_\eta) = 0$$

$$\Rightarrow c = -\frac{\nabla F(x^*_\eta)}{\eta}.$$

Need to show:

$$c^T (x^*_\eta - x^*) \le m/\eta .$$

$$\frac{\nabla F(x^*_\eta)^T}{\eta} (x^* - x^*_\eta) \le m/\eta .$$

Will prove that $\forall\ x, y \in K$:

$$\nabla F(x)^T \cdot (y - x) \le m.$$

$$\nabla F(x)^T (y - x) = \sum_{i=1}^{m} \frac{A_i}{b_i - A_i x} \cdot y - x$$

$$= \sum_{i=1}^{m} \frac{A_i y - A_i x}{b_i - A_i x}$$

$$= \sum \frac{b_i - A_i x - (b_i - A_i y)}{b_i - A_i x}$$

$$= m - \sum \frac{b_i - A_i y}{b_i - A_i x} \le m.$$

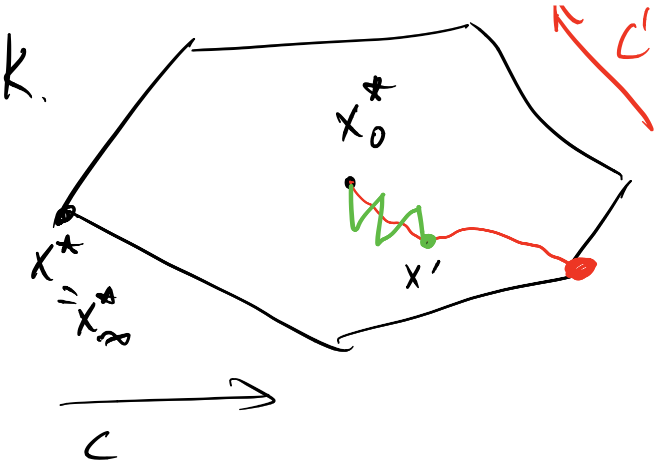
$\boxtimes$

<u>Starting point</u> x^*_0.

Can compute $x^*_{\eta_0}$ from x^*_0 by N's m.

<u>Step 1</u>: suppose we have $x' \in K$.

Claim: $\forall x' \in K \setminus \partial K$.
$\exists \eta, c'$ s.t.
$x' = \underset{x}{\operatorname{argmin}}\ \eta c' x + F(x)$



pf:
$$\nabla(\eta c' x + F(x))(x') = 0$$
$$\eta c' + \nabla F(x') = 0$$
$$c' = -\frac{\nabla F(x')}{\eta}.$$
fix $\eta = 1$ $\boxtimes$

Algo: 1) given x', compute $c' = -\nabla F(x')$
$\eta = 1$.

2) walk the central path back

decrease $\eta_{t+1} = \eta_t (1-\alpha)$.

take a ~~N~~'s step.

3) stop at $t = T$ large enough
so that we are close to x_0^*

Remark: finding a feasible x':

solve LP': $\min t$

s.t. $A_i x \le b_i + t$.

LP': a feasible sol:

$x = 0$.

$t = + \max_i -b_i$.

Remark: enough to set $\alpha = \theta\left(\frac{1}{\sqrt{m}}\right)$.