AA Lecture 19 3/23/21

Linear Programming Duality

Standard form: $\boxed{\begin{aligned} v^* = \min\ & c^T x \\ \text{s.t.}\ & Ax = b \\ & x \geq 0 \end{aligned}}$ $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$.

(primal)

1) if want to prove $v^* \leq V$
just show some $x \in F$, $c^T x \leq V$.

2) $v^* \geq \ldots$?

Def: dual of s.f. LP: $\boxed{\begin{aligned} w^* = \max\ & b^T y \\ \text{s.t.}\ & A^T y \leq c \end{aligned}}$

$y \in \mathbb{R}^m$.

Thm [weak duality]: $\boxed{w^* \leq v^*}$.

pf: fix $x \in F$: $Ax = b$ and $x \geq 0$.

fix any $y \in \mathbb{R}^m$ $(y^T A) x = y^T b$

a vector $\in \mathbb{R}^n$.

Suppose y s.t. $y^T A \leq c^T$ (each coord.)

$$\Rightarrow \quad y^T b = y^T A x \le c^T x$$

$$\Rightarrow \quad \underbrace{\max_{y:\, y^T A \le c^T} y^T b}_{= w^*} \;\le\; \underbrace{\min_{\substack{x:\, Ax = b \\ x \ge 0}} c^T x}_{= v^*}$$

☒

Dual of the Dual:

Primal:

$$\boxed{\begin{aligned} v^* = \min \ & c^T x \\ \text{s.t.} \ & Ax = b \\ & x \ge 0 \end{aligned}} \quad \Rightarrow$$

Dual:

$$\boxed{\begin{aligned} w^* = \max \ & b^T y \\ \text{s.t.} \ & A^T y \le c \end{aligned}}$$

Dual is standard form:

$$-w^* = \min_{A^T y \le c} -b^T y = \min -b^T (y^+ - y^-)$$

$$A^T(y^+ - y^-) + \delta = c \qquad y^+, y^- \in \mathbb{R}^m$$

$$y^+, y^-, \delta \ge 0 \qquad \delta \in \mathbb{R}^n$$

$$\begin{array}{ccc} & & \\ \left[\begin{array}{c|c|c} A^T & -A^T & I_n \end{array}\right] \\ y^+ \quad\; y^- \quad\; \delta \end{array}$$

Dual of the dual:

unknowns: $z \in \mathbb{R}^n$

objective func: $\max \ c^T z$

Constraints:

$$\begin{bmatrix} A \\ -A \\ I \end{bmatrix} \cdot z \le \begin{bmatrix} -b \\ +b \\ 0 \end{bmatrix}$$

$$A_i^T z = b_i \Leftarrow \begin{cases} A_i^T \cdot z \le -b_i \\ -A_i^T \cdot z \le +b_i \\ z_i \le 0. \end{cases}$$

$A_i = i^{th}$ row of A

$z' = -z.$

Dual of (-Dual): $\max\ c^T(-z')$ s.t. $Az' = b$, $z' \ge 0$

Dual of the Dual: $-\max -c^T z' = \min c^T z'$ s.t. $Az' = b$, $z' \ge 0.$

= Primal.

Strong duality:

Thm: $u^* = v^*.$

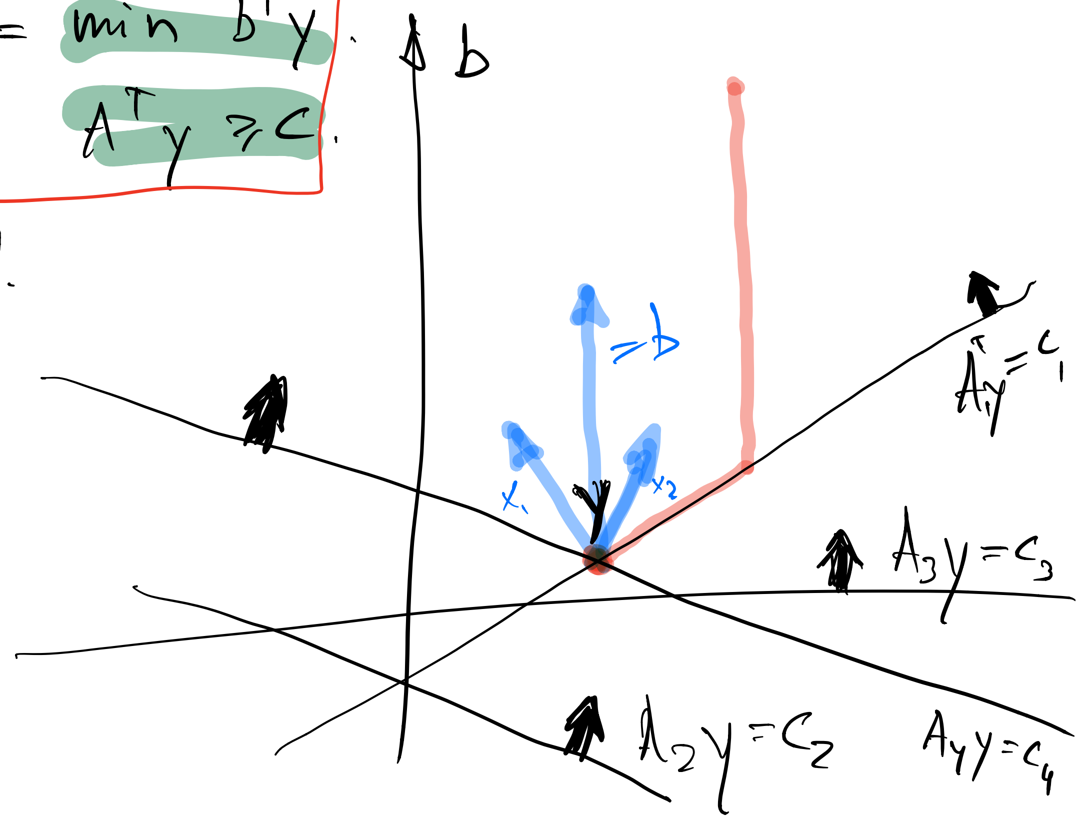
$$\boxed{\begin{aligned} v^* &= \min\ c^T x \quad Ax = b,\ x \ge 0 \\ u^* &= \max\ b^T y \quad A^T y \le c. \end{aligned}}$$

Intuition:

Consider dual: with $y \to -y$
$\qquad\qquad\qquad\qquad\quad c \to -c$

$$\boxed{\begin{aligned} -w^* = \ & \min \ b^T y \\ \text{s.t.} \ & A^T y \geq c \end{aligned}}$$

$y \in \mathbb{R}^m$.



What does it mean that ball is <u>resting</u>?

=> a set S of constraints/half-spaces:

1) $i \in S \Rightarrow A_i y = c_i$ (tight).

2) $\exists$ coefficients $x_1 \dots x_n \in \mathbb{R}_+$

$$\sum x_i A_i = b. \iff Ax = b.$$

2') if $i \notin S$, $x_i = 0$.

Value of primal: $-c^T x$

dual: $-b^T y$.

$$\sum x_i A_i y = y^T A x \overset{①}{=} \sum x_i c_i = c^T x.$$
$$\overset{②}{=} y^T b = b^T y.$$

$\boxtimes$

Note: for every $i \in [n]$: must have

$x_i = 0$, or

$A_i y = c_i$ (or both).

(complementary slackness).

Ellipsoid Algorithm: [Khachiyan '79].

Solves feasibility problem.

LP: $\overset{v^*}{\min} c^T x$

$Ax \geq b$.

$$Q_v = \{x : Ax \geq b, \ c^T x \leq v\}.$$

FP: find $x \in Q_v$ if exists.

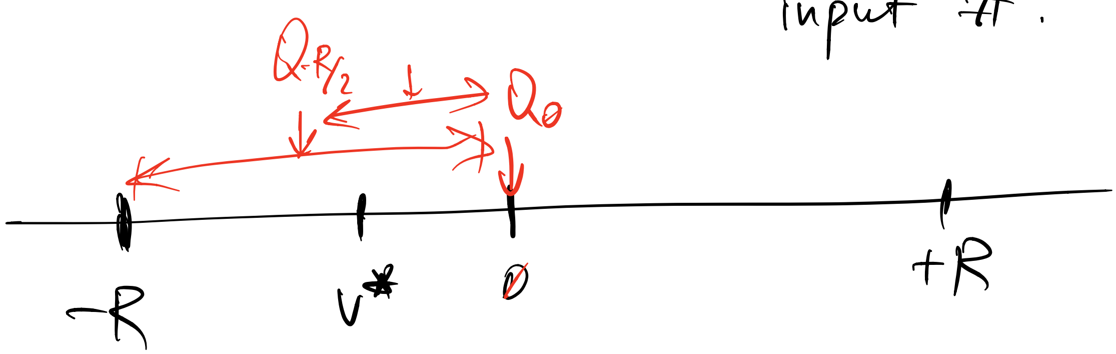
Remark: can solve LP using FP with poly-time slow-down.

pf: binary search on $\underline{v}$.

let R = upper bound on value of $|v^*|$.

$$\lg_2 R \le \text{poly}(n, m) \cdot B$$

B: # bits to represent each input #.



BS: fix $\ell = -R$, $r = +R$

$$m = \frac{\ell + r}{2}$$

check if Q_m is feasible

if yes $\Rightarrow$ recurse on $\ell = \ell$, $r = m$.

if no $\Rightarrow$ recurse on $\ell = m$, $r = r$.

stop when $r - \ell$ is sufficiently small, $\approx \frac{1}{R^{O(1)}}$.

(then opt. sol. $v^* \approx \ell \approx r$.)

$\#$ BS iterations $= O(\log R) = \underline{O(\mathrm{poly}(n,m)B)}$.

<u>Solving Q_v</u> $= \left\{ x : \begin{array}{l} Ax \geq b \\ c^T x \leq v \end{array} \right\}$.