AA Lecture 18 3/18/21

Solving system of linear equations

$Ax = b$.

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \end{array} \cdot \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} \\ b \\ \\ \end{bmatrix}$$

When $n \neq m$ or $\det(A) = 0$:

S = set of col's from A lin. indep.

$= \{s_1, \dots, s_k\} \subset \mathbb{R}^m$

$\bar{S}$ = completion to basis in $\mathbb{R}^m$

solve lin. sys.

$$\begin{array}{cc} m & \begin{bmatrix} s_1 & s_2 & \dots & s_k & \bar{s}_{k+1} & \dots & \bar{s}_m \end{bmatrix} \\ & m \end{array} \cdot \begin{bmatrix} x' \\ y \end{bmatrix} = b \qquad \text{dim. } k$$

$\Rightarrow \exists$ unique sol (x', y) sat

since $b \in \operatorname{span}(S) \Rightarrow \exists x'$ s.t.

$$\begin{bmatrix} s_1 & s_2 & \dots & s_k \end{bmatrix} \cdot \begin{bmatrix} x' \end{bmatrix} = b$$

$\Rightarrow y = 0$ (otherwise contrib of y

$\neq 0$)

$$\Rightarrow \quad x_i = \begin{cases} x'_i, & \text{if } s_i \text{ is } i\text{th col of } A \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$A \cdot x = \begin{bmatrix} \cdots & s_1 & \cdots & s_2 & \cdots \end{bmatrix} \cdot \begin{bmatrix} x \end{bmatrix} = b.$$

$\boxtimes$

No solution to $Ax = b$.

witness/certificate that no solution.

Claim (proto-Farkas lemma):

$Ax = b$ has no sol $\Longleftrightarrow$ $\exists y \in \mathbb{R}^m$ s.t.

$y^T A = 0$ and

$y^T b \neq 0.$ $\Leftrightarrow y^T b = 1.$

proof: $\boxed{\Leftarrow}$

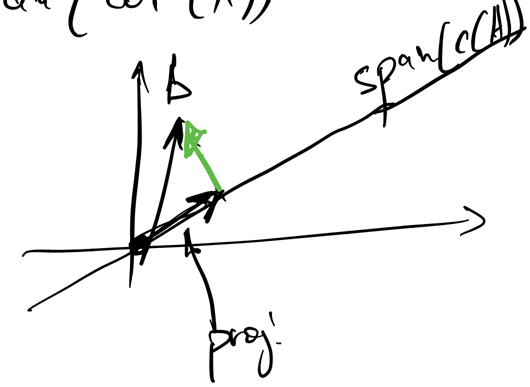
By contradiction: suppose $\exists x \in \mathbb{R}^n$ s.t. $Ax = b$

$$\underset{\substack{\| \\ 0 = 0 \cdot x}}{y^T A x} = \underset{\substack{\not\| \\ 0}}{y^T b} \quad \text{— contradiction.}$$

$\boxed{\Rightarrow}$ since $Ax = b$ has no sol

$\Rightarrow \quad b \notin \{ Ax : x \in \mathbb{R}^n \} = \text{span}(\text{col}(A))$.

$\text{proj}_A b \triangleq$ projection of b onto space span(col(A))



$y \triangleq b - \text{proj}_A b$.

$\Rightarrow \quad y \perp \text{span}(\text{col}(A))$

$\Rightarrow \quad y^T \cdot (\text{each col of } A) = 0$

$y^T A = 0$.

$y^T \cdot b = y^T \cdot (y + \text{proj}_A b) = \|y\|^2 + 0 \neq 0$. $\boxtimes$

Note: to find y:

$$\left.\begin{aligned} y^T A = 0 &\Leftrightarrow A^T y = 0 \\ b^T y &= 1 \end{aligned}\right\}$$

$$\begin{bmatrix} A^T \\ b^T \end{bmatrix} \cdot \begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Back to LP.

General form LP:

$$
\begin{aligned}
&\min \quad c^T x \\
&\text{s.t.} \quad Ax \geq b
\end{aligned}
$$

Standard form LP:

$$
\begin{aligned}
&\max \quad c^T x \\
&\text{s.t.} \quad Ax = b \\
&\qquad\quad x \geq 0. \\
&x \in \mathbb{R}^n
\end{aligned}
$$

<u>Statement</u>: these are equivalent

$\min \to \max$ by considering $-\min(\underline{-c^T})\cdot x$

① $x \in \mathbb{R} \longrightarrow x^+, x^- \geq 0$

$x = x^+ - x^-$

$$Ax \geq b \iff A(x^+ - x^-) \geq b$$
$$x^+, x^- \geq 0.$$
$$\Downarrow$$
$$\begin{bmatrix} A & -A \end{bmatrix} \cdot \begin{bmatrix} x^+ \\ x^- \end{bmatrix} \geq b.$$

② for each constraint:

$$\begin{bmatrix} A_i & -A_i \end{bmatrix} \cdot \begin{bmatrix} x^+ \\ x^- \end{bmatrix} \geq b_i$$

$(A_i = i^{th}$ row$)$

add a new var $\boxed{\xi_i \geq 0.}$

$$A_i \cdot (x^+ - x^-) \geq b_i \iff \begin{cases} A_i(x^+ - x^-) - \xi_i = b_i \\ \xi_i \geq 0. \end{cases}$$

slack var's.

1+2: $\max \; c^T(x^+ - x^-)$

s.t. $A(x^+ - x^-) - \xi = b$

$x^+, x^-, \xi \geq 0.$

Basic def's for LP

$Ax = b$

$x \geq 0.$

Standard form.

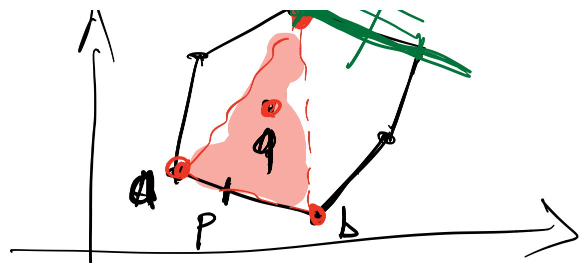
Def: an ineq. is tight if "=" holds.

Def: $x \in F = \{x ; Ax = b, x \geq 0\}$ is basic feasible solution (bfs) iff it is not a convex combination of other y's in F:

$\nexists \; y^1, \dots, y^{n+1} \in F, \; \alpha_1, \dots, \alpha_{n+1} \in \mathbb{R}_+$ s.t.

$$x = \sum_{i=1}^{n+1} \alpha_i \cdot y^{i+1} \quad \text{and} \quad \sum \alpha_i = 1.$$

(and $y^1, \dots, y^{n+1} \neq x$).



Fact: basic feasible solutions are vertices of polytope F.

Claim: if LP solution x^* is feasible and bounded ($\neq$ infinity), then there exists an opt. sol that is b.f.s.

pf: x^* if is not bfs.

$\Rightarrow$ x^* has $\leq n-1$ tight l.i. constraints, the other ones are not tight.

C = space of points defined

$\ni$
x^*

$\Rightarrow$ $\dim C \geq 1$

$\Rightarrow$ $\exists$ direction $d \in \mathbb{R}^n$, $d \neq 0$

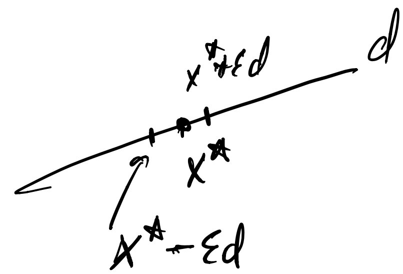
$x^* + \alpha d \in C$, $\forall \alpha \in \mathbb{R}$.

consider $\varepsilon > 0$ small:

$x^* \pm \varepsilon d \in C \leftarrow$ all tight constraints on x^*.

all others $x_i > 0$. (constraints of LP)

$\Rightarrow \exists$ small enough $\varepsilon > 0$ s.t. $x^* \pm \varepsilon d$ is still feasible.



$c^T(x^* \pm \varepsilon d)$

$= c^T x^* \pm \varepsilon c^T d$ $\neq 0$

Can consider either $x^* \pm \varepsilon d$.

One of them decreases some coord of x^*.

Push as much as allowed

$\Rightarrow$ some coord of $x^* \pm \varepsilon d$ becomes 0.

$\Rightarrow$ made one more tight constraint.

After at most $2n-1$ iterations,

will have ~~n~~ tight l.i. constr. $\boxtimes$

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x is vertex $\iff$ x is bfs

$\iff$ x has n l.i.

tight constraints.

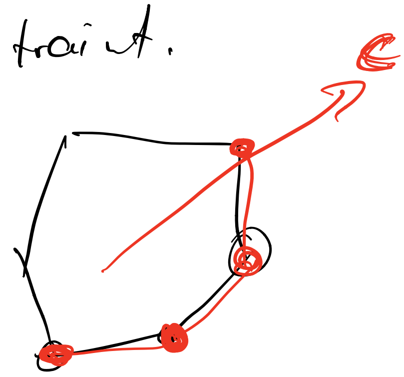
Algo 0: enumerate all vertices.

Alg 1: Simplex Algo.

1. Choose $x^0 \in F$ starting vertex.

2. $N(x^t)$ = vertices neighboring x^t.

= vertices s.t. they diff. from x^t in only 1 tight constraint.



3. Choose $y \in N(x^t)$ s.t.

$c^T y < c^T x^t$ (assuming min)

4. Set $x^{t+1} = y$, repeat until y does not exist.

pivoting rule.

<u>Note:</u> how to find starting $x^0 \in F$.

Gen: $\min c^T x$

$Ax \le b$.

solve another LP:

$$\boxed{\begin{aligned} &\min t \\ &\text{s.t. } Ax - t\cdot \mathbb{1} \le b. \end{aligned}}$$

- if opt. $t > 0 \Rightarrow$ orig LP is infeas. ($F = \emptyset$).
- if opt $t \le 0$, x = starting vertex

initialize by setting $x = 0$

$t = -\min_i b_i$.

Remark: 1) taking y^* = best improv. is not optimal.

2) Simplex Algo takes exp time for most pivoting rules we know.

3) in practice works well.

Conjecture [Hirsch '57]: for any starting vertex in polytope F, any other vertex, $\exists$ shortest path of length poly(n, m).

Optim. Conj: $\leq m - n$.
↳ Disproved in [Santos '10]! $n = 43$, $m = 86$.

Smoothed Analysis: [Spielman - Teng '00s]

instance of LP: $\min c^T x$
$Ax \leq b$.

Consider ins$'$: $c' = c +$ gaussian noise

$A', b' = A, b +$ g. n.

then Simplex on $\min c'^T x$

$A'x \leq b'$

runs in poly time.