AA 3/9/21

Thm: $\lambda_2 = 1$ iff G is disconnected.

$\hat{A} = D^{-1/2} A D^{-1/2}$ with eigenv's $\lambda_1 \geq \lambda_2 \geq \dots \lambda_n$
$v_1 \dots v_n$.

Def: Laplacian of G:

$$L_G = D - A = \begin{array}{cc} & j \\ i & \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & -1 \\ & & & d_n \end{bmatrix} \end{array}$$

edge (i,j).

Fact: $\forall x \in \mathbb{R}^n$:

$$\boxed{x^T L x = \sum_{(i,j) \in E} (x_i - x_j)^2.}$$

Pf: $L = \sum_{e \in E} L_e$, $L_e = \begin{array}{ccc} i & j & \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & & \begin{matrix} i \\ j \end{matrix} \end{array}$, $e = (i,j)$

$$x^T L x = \sum_{e \in E} x^T L_e x = \sum_{e=(i,j) \in E} (x_i^2 + x_j^2 - 2x_i x_j)$$

$$= \sum_{e=(i,j) \in E} (x_i - x_j)^2 \quad \square$$

Intuition: x = potentials @ nodes in G.

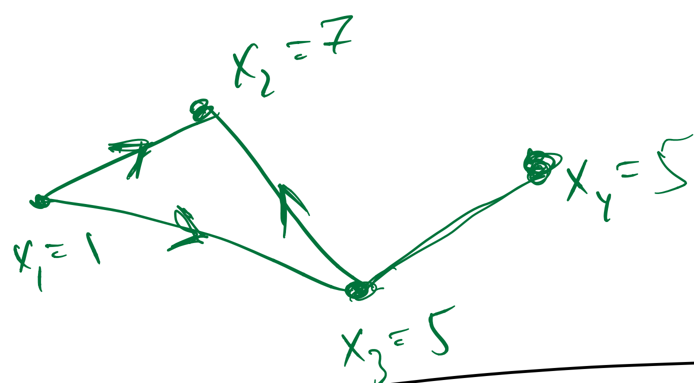
$x_i - x_j$ = potential Δ.

Energy loss per edge (i,j)

$$= I_{ij} \cdot V_{ij} = \frac{V_{ij}}{R_{ij}} \cdot V_{ij} = V_{ij}^2.$$

$$\text{Total energy loss} = \sum_{\substack{e=(i,j)\\ \in E}} (x_i - x_j)^2$$

$$= x^T L x.$$



Properties of Laplacian: $x^T L x \geq 0$

$\Rightarrow$ all eigenval's ≥ 0.

$$x = (1, 1, \dots, 1) = e.$$

$$x^T L x = \sum_{(i,j)\in E} (x_i - x_j)^2 = 0.$$

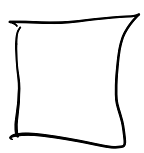
$\mu_1(L)$ = smallest eigenval of $L = 0$.

Thm: $\mu_2(L) = 0$ iff G is disconnected

Drawing a graph G in $\mathbb{R}^2$

Problem: given $G = (V, E)$, draw in $\mathbb{R}^2$.

G = graph corresponding to a square
$V = \{1, 2, 3, 4\}$, $E = \{(1,2), (2,3), (3,4), (4,1)\}$.



What is a drawing?

$f : V \to \mathbb{R}^2$ $\qquad f(i) = (x_i, y_i)$.

Spectral drawing: $f(i) = (v_{2i}, v_{3i})$

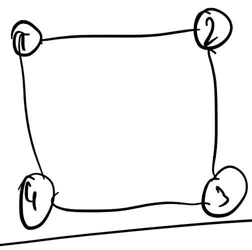
$v_2 = 2^{nd}$ eigenvector L

$v_3 = 3^{rd}$ — " — L.

$(v_1 = (1, 1, \dots, 1)$.

$Lv_1 = (D - A)e = De - Ae$

$$= (d_1, d_2, \dots d_n)$$
$$- (d_1, d_2, \dots d_n)$$
$$= (0, 0, \dots 0)$$
$$= 0 \cdot v_1$$



Let's draw in 1D.

$x \in \mathbb{R}^n \to$ coordinate per node in V.

Attempt #1: $$\min_{x \in \mathbb{R}^n} \sum_{(i,j) \in E} (x_i - x_j)^2$$

$$x = (0, 0, \dots 0)$$

A2: $$\min_{\|x\|_2 = 1} \sum_{ij \in E} (x_i - x_j)^2$$

$$x = \left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \dots \frac{1}{\sqrt{n}}\right) = 1^{st} \text{ eigenvector}$$

A3: require $\|x\|_2 = 1$ and $\sum x_i = 0$.

sol: $\arg\min x^T L x = v_2 = 2^{nd}$ eigenvec of L.

$\|x\|_2 = 1$
$e^T x = 0$

<u>To 2D:</u>

<u>A4:</u> have $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$.

$$\min_{\substack{\|x\|, \|y\| = 1 \\ e^T x = 0 \\ e^T y = 0}} \boxed{x^T L x + y^T L y} = \min_{x \dots} x^T L x + \min_{y} y^T L y.$$

$$x^T L x + y^T L y = \sum_{ij \in E} (x_i - x_j)^2 + (y_i - y_j)^2.$$

<u>sol:</u> $x = v_2$.

$y = v_2$

<u>A5 (final):</u>

$$\min_{\substack{\|x\|_2 = 1, \\ e^T x = 0, \\ \|y\| = 1,\ e^T y = 0 \\ y^T x = 0}} x^T L x + y^T L y$$

$$= \left[\min_{\substack{\|y\| = 1,\ e^T y = 0 \\ y \perp x = v_2}} y^T L y\right] + v_2^T L v_2.$$

$$= \min_{\substack{\|y\|=1,\\ e^T y = 0\\ v_2^T \cdot y = 0}} R_L(y) + v_2^T L v_2$$

using Raleigh quotient connection to e'vectors

$$= v_3^T L v_3 + v_2^T L v_2,$$

Sol: $X = v_2$, $y = v_3$. (or switched)

Cheeger Inequality & Spectral Clustering Partitioning.

Def: Normalized Lap: $\hat{L} = D^{-1/2} L \cdot D^{-1/2}$.

$$\hat{L} = D^{-1/2} \cdot (D - A) \cdot D^{-1/2} = I - \underbrace{\hat{A}}_{\text{norm adj.}}.$$

suppose (λ_i, v_i) are the e'val/vec of $\hat{A}$:

$$\hat{L} v_i = (I - \hat{A}) v_i = v_i - \lambda_i \cdot v_i = (1 - \lambda_i) \cdot v_i.$$

$\Rightarrow (1 - \lambda_i, v_i)$ are e'val/vec of $\hat{L}$.

$1 = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ e'val of $\hat{A}$

$0 = 1-\lambda_1 \leq 1-\lambda_2 \leq \dots \leq 1-\lambda_n$ e'val of $\hat{L}$

$0 = \mu_1 \leq \mu_2 \leq \dots \leq \mu_n \leq 2.$

<u>Thm:</u> $\mu_2 = 0 \iff G$ is disconnected.

$\mu_2 > 0 \rightarrow$ <u>algebraic connectivity.</u>

<u>Combinatorial connectivity:</u>

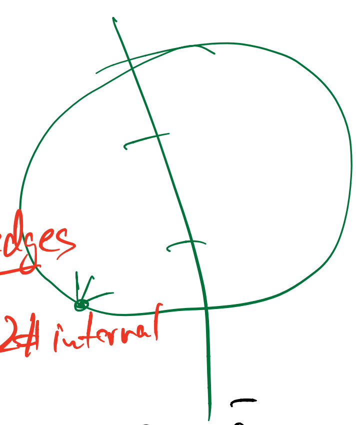
<u>Def:</u> $S \subset V$ is a cut.

$\text{vol}(S)$ = sum of degrees of $i \in S$

$= \sum_{i \in S} d_i.$

conductance of S:

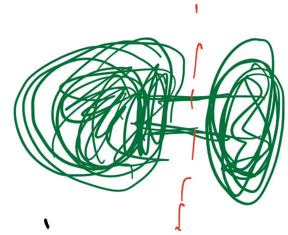
$$\phi(S) = \frac{\partial S}{\text{vol}(S)} = \frac{\#\text{cross edges}}{\#\text{cross} + 2\#\text{internal}}$$



∂S = # edges crossing $S \to \bar{S}$

$= \sum_{(i,j) \in E} \mathbb{1}[i \in S, j \notin S].$

conductance of graph G:



$$\phi(G) = \min_{\substack{S \neq \emptyset \\ \mathrm{vol}(S) \le \frac{1}{2}\mathrm{vol}(V)}} \phi(S)$$

<u>Thm</u> [Cheeger]: $\frac{\mu_2}{2} \le \phi(G) \le \sqrt{2\mu_2}$.