AA Lecture 12 2/18/21.

Max-Flow Algos.

res. graph

FF Algo:

```
$f = 0$
while $\exists$ augmenting path P in $G_f$
    augment f with P, by
        value $\delta$ = min res. cap. in P
            $= \min_{e \in P} c_e^f > 0$.
```

RT: consider graph G where all cap. are <u>integer</u>.

Claim: FF runtime is $O(|f^*| \cdot m)$, where f^* = max flow.

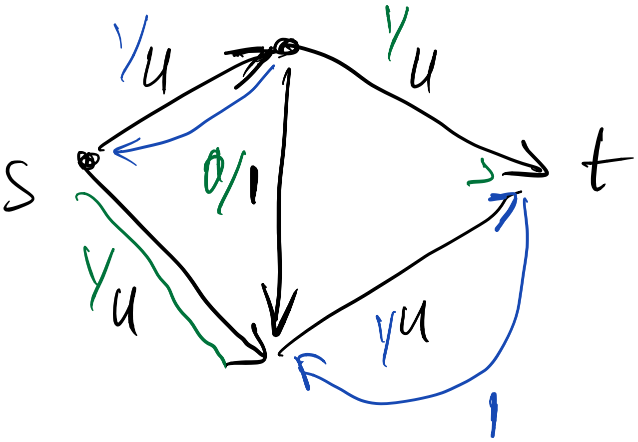
Pf: in each iteration, $\delta > 0 \Rightarrow \delta \geq 1$.

$\Rightarrow$ #iterations $\leq |f^*|$.

time/iteration $= O(n)$ by BFS/DFS.

☒

Bad case:



$u = 10^6$.

$\# iter = \Theta(u)$.

not poly-time. want time poly in $(n, m, \lg u)$.

Choice of P in G_f:

- most aug path : maximizes δ.
  max-width path.
- random path.
- shortest path. (BFS).

Max-width path:

FF where P is ~~the~~ a s-t path with maximal δ in G_f.

iterations?

suppose current flow f

remaining value $v = |f^*| - |f|$.

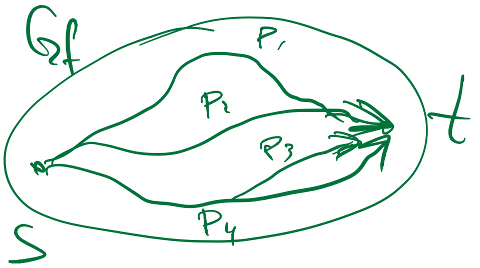
Claim: $\delta \geq v/m$.

proof: G_f it has max-flow val. v

$\Rightarrow$ the max flow φ_f^* in G_f:

$$\varphi_f^* = \sum_{i=1}^{k} f_{P_i} + \sum_{j=1}^{\ell} f_{C_j}$$

$$\Rightarrow \underset{v=}{\;} |\varphi_f^*| = \sum_{i=1}^{k} |f_{P_i}| \leq k \cdot |f_P|$$



~~p~~path flow with max δ.

$\boxtimes$

After augmenting with P (of max width δ),

value of remaining flow is $\leq v - \frac{v}{m}$

$= v \cdot (1 - 1/m)$

$\Rightarrow$ in t iterations, the remaining flow:

$$|f^*| - |f| \le \underbrace{|f^*| \cdot (1 - 1/m)^t}_{< 1}.$$

$$|f^*| \cdot (1 - 1/m)^t < 1 \quad \Rightarrow$$

$$|f^*| \cdot (1 - 1/m)^t \le m \cdot \underset{\substack{\uparrow \\ \text{max capacity}}}{U} \cdot e^{-t/m}$$

set $T = m \cdot \ln(m \cdot U) + 1$, then we get $|f^*| (1 - 1/m)^T < 1$.

so $T = O(m \cdot \ln(mU))$.

Total time: $T \cdot \boxed{[\text{time to find max-width path } P].}$

1) dynamic programming (à-la Dijkstra's)

2) do binary search on the value δ

of the max width path P:

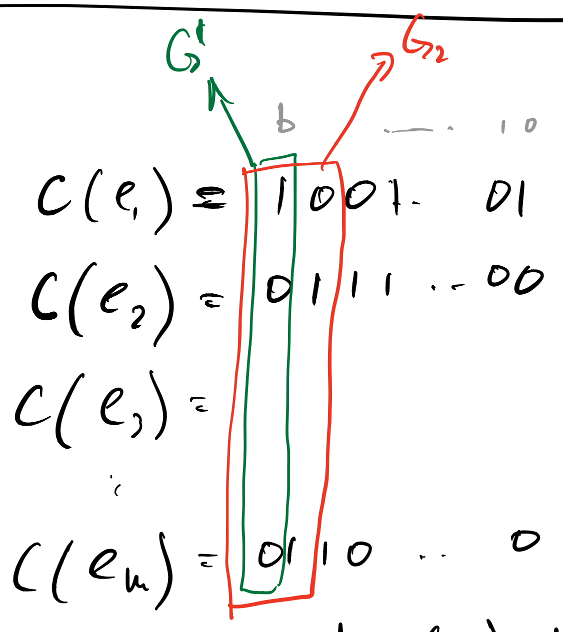
- to test whether $\exists$ path P with $\delta(P) \geq$ threshold θ: drop all edges in G_f with $c^f < \theta$. just check $\exists$ $s \to t$ path.

$\Rightarrow$ time is $O(m \cdot \lg U)$.

Total time: $O(m^2 \lg U \cdot \lg m U)$.

Algo 2: Scaling.

$b \triangleq \lceil \lg_2 U \rceil$.



G^i = graph G, where $c^i(e_j) = \left\lfloor \frac{c(e_j)}{2^{b-i}} \right\rfloor$

$G^b = G$.

$f^{-1} = 0$.

Algo: for $i = 0 \ldots b$,

find max flow $|f^i|$ in G^i,
using FF starting from flow
$\boxed{2 \cdot f^{i-1}}$.

report f^b.

Correctness:

Claim: flow $2 \cdot f^{i-1}$ is valid flow in G^i.

pf: by induction.

@ start: flow $= 0$ by initialization

inductive step:

by IH: $0 \le f^{i-1}(e) \le c^{i-1}(e)$

and f^{i-1} sat. flow conserv.

$$\Rightarrow 0 \le 2 f^{i-1}(e) \le 2c^{i-1}(e) \le c^i(e).$$

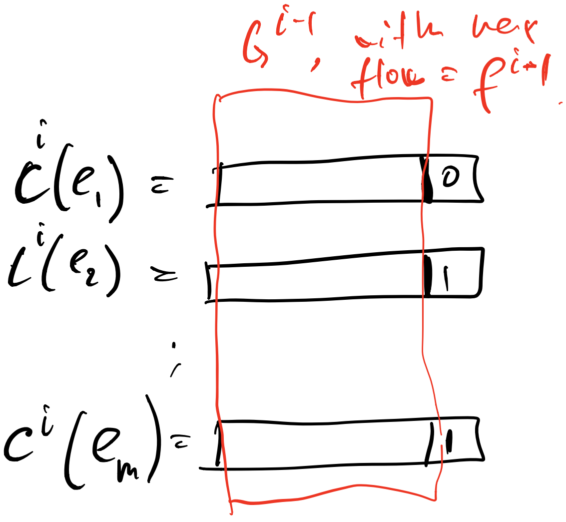
flow conservation is imm. ☒

Runtime: b scaling stages $\times$ $\boxed{\text{time to run FF in } G^i \text{ starting from } 2f^{i-1}.}$

upper-bounded by max-flow in the residual graph $G^i_{2f^{i-1}}$. ($\times m$).

Claim: remaining flow in $G^i_{2f^{i-1}}$ is $\leq m$.

Pf: $2f^{i-1}$ is max flow in the graph with cap. $c'(e)=2c^{i-1}(e)$



$$c^i(e) \in \{c'(e), c'(e)+1\}.$$

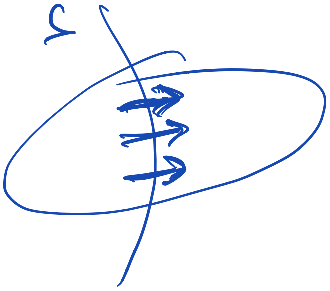
Use max-flow-min-cut thm:

1) $\forall f, \forall S:$

$$|f| \leq c(S)$$

2) $\max_f |f| = \min_S c(S).$

$$\xrightarrow[\text{②} \text{ on } G^{i-1}]{} \exists \text{ } s\text{-}t \text{ cut } S \text{ s.t.}$$



$$|f^{i-1}| = c^{i-1}(S).$$

$$c^i(S) \leq 2 \cdot c^{i-1}(S) + \left[\begin{matrix}\#\text{edges} \\ S \to \bar{S}\end{matrix}\right]$$

$$\leq 2 \cdot |f^{i-1}| + m$$

$$\overset{①}{\Rightarrow} |f^i| \leq 2 \cdot |f^{i-1}| + m$$

$$\Rightarrow \underbrace{|f^i| - |2 \cdot f^{i-1}|}_{\text{remaining flow in graph } G^i \text{ when starting } 2f^{i-1} \text{ flow.}} \leq m. \quad \boxtimes$$

Time: $O(\lg U \cdot m \cdot m) = O(m^2 \cdot \lg U).$

Algo 3: use FF with P = shortest path in G^f.

Real RAM model: all registers / cells in mem contain reals. Operations:
- add/deletion, multipl./div., max/min.

RT? $O(m^2 n)$. time total.

Def: $d_f(u,v) = d_{G_f}(u,v)$ = shortest path in G_f.

Claim 1: fix flow f. let P = augmenting path.

$f' = f$ augm. with P.

$$d_{f'}(s,v) \geq d_f(s,v).$$

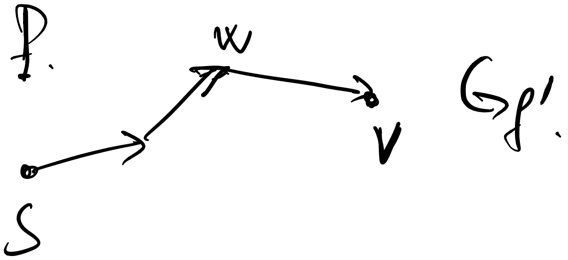
pf: by contradiction:

$$A \triangleq \{v : d_{f'}(s,v) < d_f(s,v)\} \neq \emptyset.$$

fix $v = \min_{v \in A} d_{f'}(s,v)$.

Consider shortest path P in $G_{f'}$.
from s to v.

u = preceding node in P.

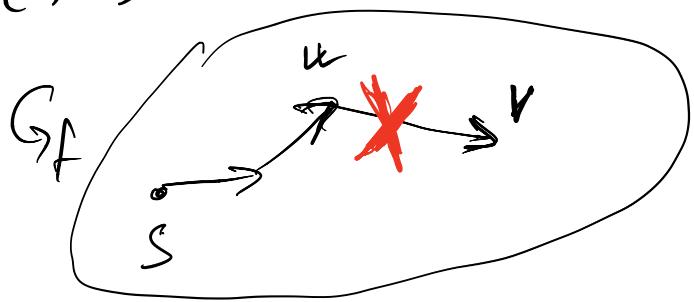


$\Rightarrow d_{f'}(s,u) \geq d_f(s,u)$ (since v is min in A).

$$\underset{\underset{d_f(s,v)}{\wedge}}{d_{f'}(s,v)} = d_{f'}(s,u)+1 \geq d_f(s,u)+1.$$

$\Rightarrow d_f(s,v) > d_f(s,u)+1.$

$\Rightarrow (u,v) \notin G_f.$



$\Rightarrow$ since $(u,v) \in G_{f'}$
shortest aug. path $\overset{P}{\ }$ in G_f has to pass $v \to u$ $\Rightarrow$

$$d_f(s,u) = d_f(s,v) + 1 \geq d_f(s,u) + 2$$

contradiction! $\boxtimes$