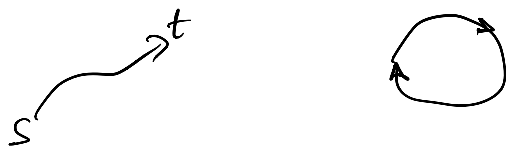
AA Lecture 11 2/16/21

Max Flow problem

find max $f \in \mathbb{R}_+^m$ in $G = (V, E, c)$.

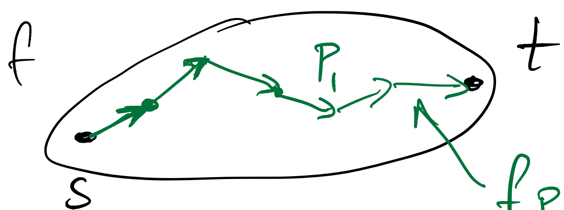
Path Flow Decom. Thm: any flow f,

$$f = \sum_{i=1}^{k} f_{P_i} + \sum_{j=1}^{e} f_{C_j}$$



Proof:

1)



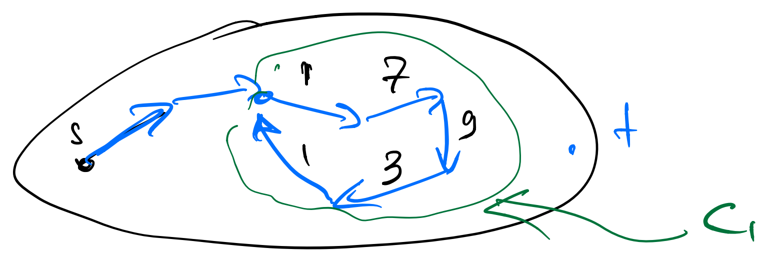
f_{P_1} : on edges S,

$$f_{P_1} = \min_{e \in S} f_e \cdot (0, 0, 1, \ldots\ 0, 1)$$

1's in $e \in S$.

$$|f_{P_1}| = \min_{e \in S} f_e > 0.$$

2)

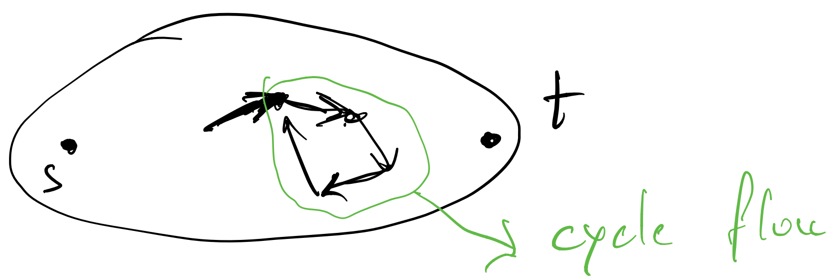


$$f_{C_1} = \min_{e \in G} f_e \cdot (0\ 1, \ldots\ 0)$$

Is in $e \in G$.

$$f' = f - f_{P_1} \iff f = f_{P_1} + f'.$$

3)



Convergence: each time we extract f_{P_i} or f_{C_j}, some edge $f_e > 0 \to f_e = 0$.

$\Rightarrow k + \ell \leq m.$ $\boxtimes$

Algorithms:

$f^* =$ max flow.

$f^* = \sum f^*_{P_i} + \sum f^*_{C_j}.$

Alg $\emptyset$:

if $|f^*| > 0$, then exists some f_P, $|f_P| > 0$. can find it by finding any path

$s \to t$ on edges e with $c_e > 0$

Eg: use BFS/DFS for ~~r~~reachability

$s \to t$ on edges $c_e > 0$.

Alg 0.5: find such path iteratively.

$f = 0$

while we can:

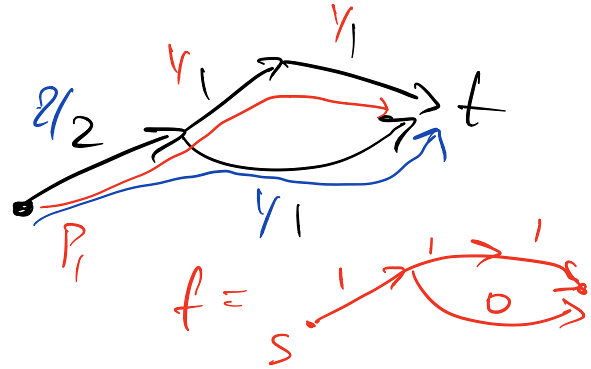
find a path P $s \to t$ st:

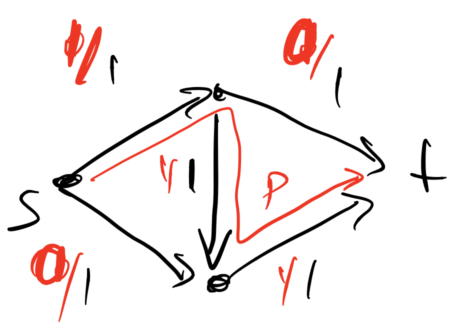
$\forall e \in P$, $\underbrace{c_e - f_e}_{\text{residual capacity}} > 0$

$f' := f + \mathbb{1}_P \cdot \min_{e \in P} c_e - f_e$.

vector with 1s in $e \in P$.

$f := f'$.





$|f^*| = 2$

$|f_p| = 1.$

Alg 1: (Ford - Fulkerson):

idea: route flow on edges

Def: residual graph: $G_f = (V, E', c^f)$

for every $e \in E$, $e = (u,v)$:

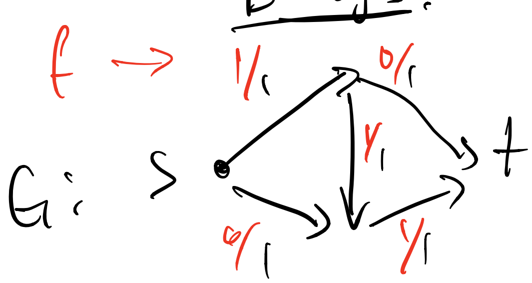
- if $c_{uv} > f_{uv}$, then add edge (u,v) to E', $c^f_{u,v} = c_{uv} - f_{uv}$.

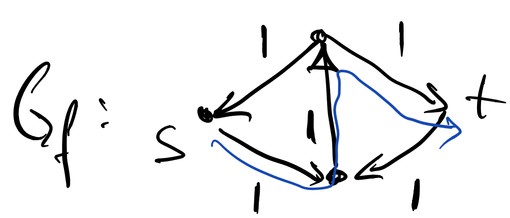
F-edge.

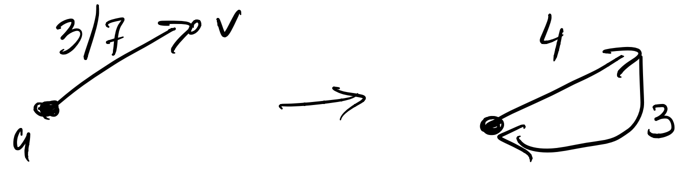
- if $f_{uv} > 0$, then add edge (v,u)

$$c^f_{vu} = f_{uv}.$$

B-edges.







FF Algo: $f := 0$

$G_f := G$.

while $\exists$ path $s \to t$ in G_f:

- $P =$ path $s \to t$ in G_f.
- $\delta = \min_{e \in P} c_e^f > 0$
- for $e \in P$:
  - if e is F-edge
    - $f_e := f_e + \delta$.
  - else ($e = (u,v)$ is B-edge)
    - $f_{vu} := f_{vu} - \delta$.
- Build new G^f.

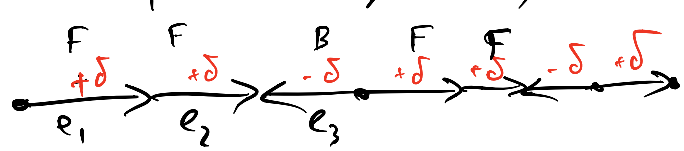
augmenting path

Claim 1: flow f is valid

Claim 2: flow f is maximum.

Claim 3: how many iterations? (R.T.)

Proof Cl 1: fix f, P, $\delta = \min_{e \in P} c_e^f > 0$.



F: $\delta \le c^f_{e_1} = c_{e_1} - f_{e_1}$

$\Rightarrow f_{e_1} + \delta \le c_{e_1}$.

B: $e_3 = (v,u)$, $c^f_{u,v} = f_{vu} > 0$

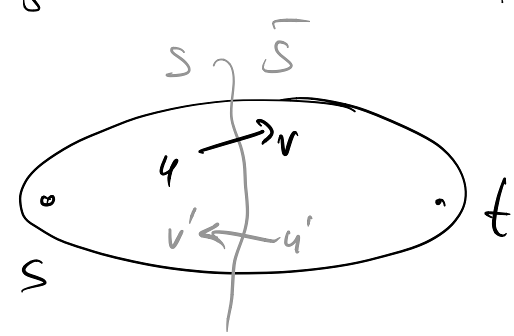
$\delta \le c^f_{u,v} = f_{vu}$.

$\Rightarrow f_{e_3} - \delta \ge f_{e_3} - f_{vu} = 0$.

Claim 2 proof:

Def: s-t cut is $(S, \underbrace{V \setminus S}_{\bar{S}})$ s.t. $s \in S$, $t \in \bar{S}$.

$$c(S,\bar{S}) = c(S) \triangleq \sum_{\substack{u \in S \\ v \in \bar{S}}} c_{uv}.$$



Related problem: min s-t cut: find s-t cut $(S, \bar{S})$ minimizing $c(S)$.

Thm {max-flow min cut duality}: $\forall G, s, t$:

max$^{s\text{-}t}$ flow in G = min$^{s\text{-}t}$ cut in G.

$\Downarrow$

$$\boxed{\begin{array}{l}\forall \text{ flow } f \;\&\; \text{cut } S: \\ \qquad |f| \leq c(S).\end{array}}$$

Thm: these 3 statement equiv: $\forall f$:

1) f is a max flow in G.

2) G^f (res. graph) has no $s \to t$ path.

3) $\exists$ cut $(S, \bar{S})$ s.t. $|f| = c(S)$.

Note: implies Claim 2.

proof: 1 => 2: by contradiction:

if $\exists P$ $s \to t$ in G^f, then can augment with f_P:

$f' = f$ + augmenting along P

$|f'| = |f| + \delta$ $\hookrightarrow = \min\limits_{e \in P} c_e^f > 0.$

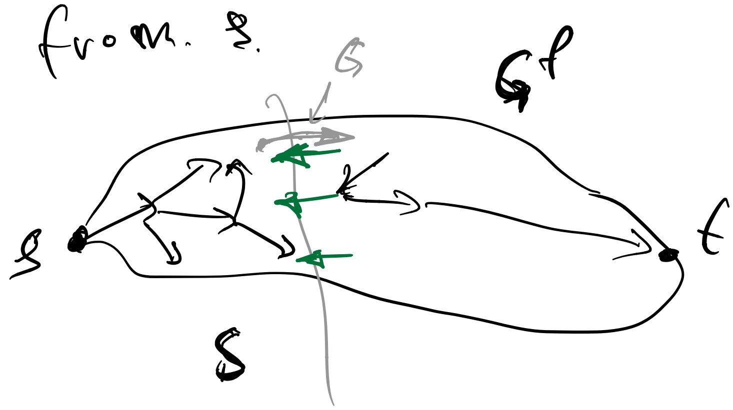
$> |f|$ contradiction.

$\underline{2 \Rightarrow 3}$: G_f no path $s \longrightarrow t$.

S = set or nodes reachable from s.

$\Rightarrow s \in S$

$t \notin S$.



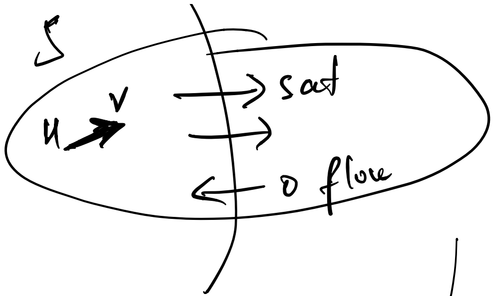
$$c(S) = \sum_{\substack{u \in S \\ v \notin S}} c_{uv} \qquad c_{uv} = f_{uv}$$

$$= \sum_{\substack{u \in S \\ v \notin S}} f_{uv}.$$

Also $\forall$ edge $(u,v) \in E$, $u \notin S$, $v \in S$

$f_{uv} = 0$ otherwise $c^f_{vu} > 0$

(contradiction to def. S)



sum up flow conservation for all nodes $u \in S$.

$$\begin{cases} \sum_{su} f_{su} - \sum_{us} f_{us} = |f| \\ \sum_{\substack{u:\,vu \\ \in E}} f_{vu} - \sum_{\substack{u:\,uv \\ \in E}} f_{uv} = 0, \quad \forall v \neq s, t \end{cases}$$

sum for all $u \in S$

$$\Rightarrow \underbrace{\sum_{\substack{v \in S \\ u \notin S \\ vu \in E}} f_{vu}}_{= c(S)} - \underbrace{\sum_{\substack{v \in S \\ u \notin S \\ uv \in E}} f_{uv}}_{= 0} = |f|.$$

In general:

$f_{vu} \leq c_{vu}$

$f_{uv} \geq 0$

$\Rightarrow$

$$|f| \leq \sum_{\substack{v \in S \\ u \notin S \\ vu \in E}} c_{vu} = c(S).$$

for any st cut S!

$$\Rightarrow c(S) = |f|.$$

<u>3 $\Rightarrow$ 1:</u> above proof can be adapted to prove

$|f| \leq c(S)$. $\forall f$ and $\forall S$.

(see

if $|f| = c(S)$ for some S

$\Rightarrow$ f is maximum. $\boxtimes$

<u>RT</u>: how many iterations FF requires?

how to choose aug. paths that most efficient.