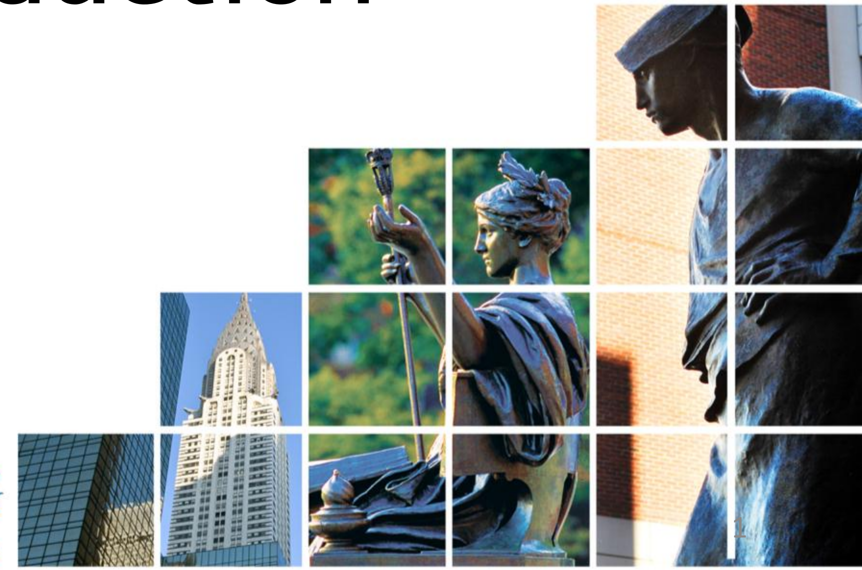
Lecture 7: Dynamic sampling

Dimension Reduction



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Plan

- Admin:
  - PSet 2 released later today, due next Wed
  - Alex office hours: Tue 2:30-4:30
- Plan:
  - Dynamic streaming graph algorithms
  - **S2**: Dimension Reduction & Sketching
- Scriber?

Sub-Problem: dynamic sampling

- Stream: general updates to a vector $x \in \{-1,0,1\}^n$

- Goal:
  - Output i with probability $\frac{|x_i|}{\sum_j |x_j|}$

Dynamic Sampling

- Goal: output i with probability $\frac{|x_i|}{\sum_j |x_j|}$
- Let $D = \{i \text{ s.t.} x_i \neq 0\}$
- Intuition:
  - Suppose $|D| = 10$
    - How can we sample i with $x_i \neq 0$?
    - Each $x_i \neq 0$ is a $1/10$-heavy hitter
    - Use CountSketch $\Rightarrow$ recover all of them
    - $O(\log n)$ space total
  - Suppose $|D| = 10\sqrt{n}$
    - Downsample: pick a random set $I \subset [n]$ s.t. $\Pr[i \in I] = \frac{1}{\sqrt{n}}$
    - Focus on substream on $i \in I$ only (ignore the rest)
    - What's $|D \cap I|$?
      - In expectation = 10
    - Use CountSketch on the downsampled stream I…
  - In general: prepare for all levels

Basic Sketch

- Hash function $g: [n] \to [n]$
- Let $h(i) = $ #tail zeros in $g(i)$
  - $\Pr[h(i) = j] = 2^{-j-1}$ for $j = 0..L-1$ and $L = \log_2 n$
- Partition stream into substreams $I_0, I_1, \dots I_L$
  - Substream I_j focuses on elements with $h(i) = j$
  - $E\left[|D \cap I_j|\right] = |D| \cdot 2^{-j-1}$
- Sketch: for each $j = 0, \dots L$,
  - Store CS_j: CountSketch for $\phi = 0.01$
  - Store DC_j: distinct count sketch for approx=1.1
    - F_2 would be sufficient here!
  - Both for success probability $1 - 1/n$

Estimation

- Find a substream I_j s.t. DC_j output $\in [1,20]$
  - If no such stream, then FAIL
- Recover all $i \in I_j$ with $x_i \neq 0$ (using CS_j)
- Pick any of them at random

```
Algorithm DynSampleBasic:

Initialize:
  hash function g: [n] → [n]
  h(i) = # tail zeros in g(i)
  CountSketch sketches CS_j, j ∈ [L]
  DistinctCount sketches DC_j, j ∈ [L]

Process(int i, real δ_i):
  Let j = h(i)
  Add (i, δ_i) to CS_j and DC_j

Estimator:
  Let j be s.t. DC_j ∈ [1,20]
  If no such j, FAIL
  i = random heavy hitter from CS_j
  Return i
```

Analysis

- If $|D| < 10$
  - then $|D \cap I_j| \in [1,10]$ for some j
- Suppose $D \geq 10$
  - Let k be such that $|D| \in [10 \cdot 2^k, 10 \cdot 2^{k+1}]$
- $E[|D \cap I_k|] = |D| \cdot \ 2^{-k-1}$ $\in [5,10]$
- $Var[|D \cap I_k|] \leq |D| \cdot 2^{-k-1} \leq 10$
- Chebyshev: $|D \cap I_k|$ deviates from expectation by $4 > \sqrt{1.5 Var}$ with probability at most $\frac{1}{1.5} < 0.7$
  - Ie., probability of FAIL is at most 0.7

```
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Estimator:
  Let j be s.t. DC_j ∈ [1,20]
  If no such j, FAIL
  i = random heavy hitter from CS_j
  Return i
```

Analysis (cont)

- Let j with $DC_j \in [1,20]$
  - All heavy hitters = $D \cap I_j$
  - CS_j will recover a heavy hitter, i.e., $i \in D \cap I_j$
- By symmetry, once we output some i, it is random over D
- Randomness?
  - We just used Chebyshev $\Rightarrow$ pairwise g is OK !

```
Algorithm DynSampleBasic:

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   h(i) = # tail zeros in g(i)
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Process(int i, real δ_i):
   Let j = h(i)
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Estimator:
   Let j be s.t. DC_j ∈ [1,20]
   If no such j, FAIL
   i = random heavy hitter from CS_j
   Return i
```
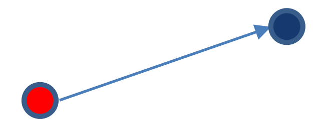

Dynamic Sampling: overall

- DynSampleBasic guarantee:
  - $\boxed{\text{FAIL: with probability} \leq 0.7}$
  - Otherwise, output a random $i \in D$
    - Modulo a negligible probability of CS/DC failing
- Reduce FAIL probability?
- DynSample-Full:
  - Take $k = O(\log n)$ independent DynSampleBasic
  - Will not FAIL in at least one with probability at least $1 - 0.7^k \geq 1 - 1/n$
  - Space: $O(\log^4 n)$ words for:
    - $k = O(\log n)$ repetitions
    - $O(\log n)$ substreams
    - $O(\log^2 n)$ for each CS_j, DC_j

Back to Dynamic Graphs

		(i,j)			
i		+1			
j		-1			



- Graph G with edges inserted/deleted
- Define node-edge incidence vectors:
  - For node v, we have vector:
    - $x_v \in R^p$ where $p = \binom{n}{2}$
    - For $j > v$: $x_v(v,j) = +1$ if edge (v,j) exists
    - For $j < v$: $x_v(j,v) = -1$ if edge (j,v) exists
- Idea:
  - Use Dynamic-Sample-Full to sample an edge from each vertex v
  - Collapse edges
  - How to iterate?
- Property:
  - For a set Q of nodes
  - Consider: $\sum_{v \in Q} x_v$
  - Claim: has non-zero in coordinate (i,j) iff edge (i,j) crosses from Q to outside (i.e., $|Q \cap \{i,j\}| = 1$)
- Sketch enough for: for any set Q, can sample an edge from Q !

Dynamic Connectivity

$x_v \in R^p$ where $p = \binom{n}{2}$
for $j > v$: $x_v(v,j) = +1$ if $\exists(v,j)$
for $j < v$: $x_v(j,v) = -1$ if $\exists(j,v)$

- Sketching algorithm:
  - Dynamic-Sample-Full for each x_v
- Check connectivity:
  - Sample an edge from each node v
  - Contract all sampled edges
  - $\Rightarrow$ partitioned the graph into a bunch of components $Q_1, \dots Q_l$ (each is connected)
  - Iterate on the components $Q_1, \dots Q_l$
- How many iterations?
  - $O(\log n)$ – each time we reduce the number of components by a factor ≥ 2
- Issue: iterations not independent!
  - Can use a fresh Dynamic-Sampling-Full for each of the $O(\log n)$ iterations

A little history

- [Ahn-Guha-McGregor'12]: the above streaming algorithm
  - Overall $O(n \cdot \log^4 n)$ space
- [Kapron-King-Mountjoy'13]:
  - Data structure for maintaining graph connectivity under edge inserts/deletes
    - First algorithm with $(\log n)^{O(1)}$ time for update/connectivity !
    - Open since '80s

Section 2:

Dimension Reduction & Sketching

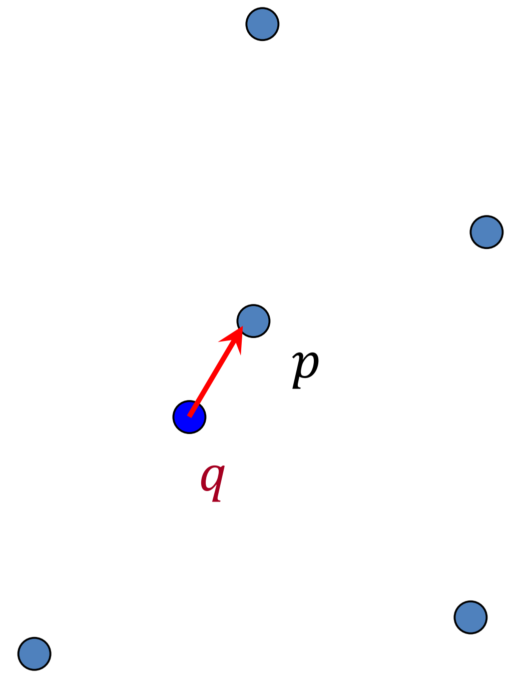
Why?

- Application:

  Nearest Neighbor Search

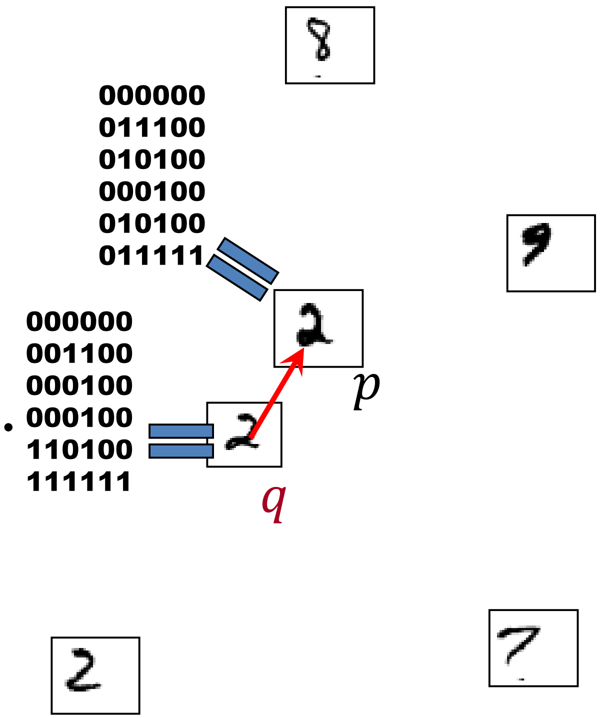
  in high dimensions

- Preprocess: a set D of points
- Query: given a query point q, report a point $p \in D$ with the smallest distance to q



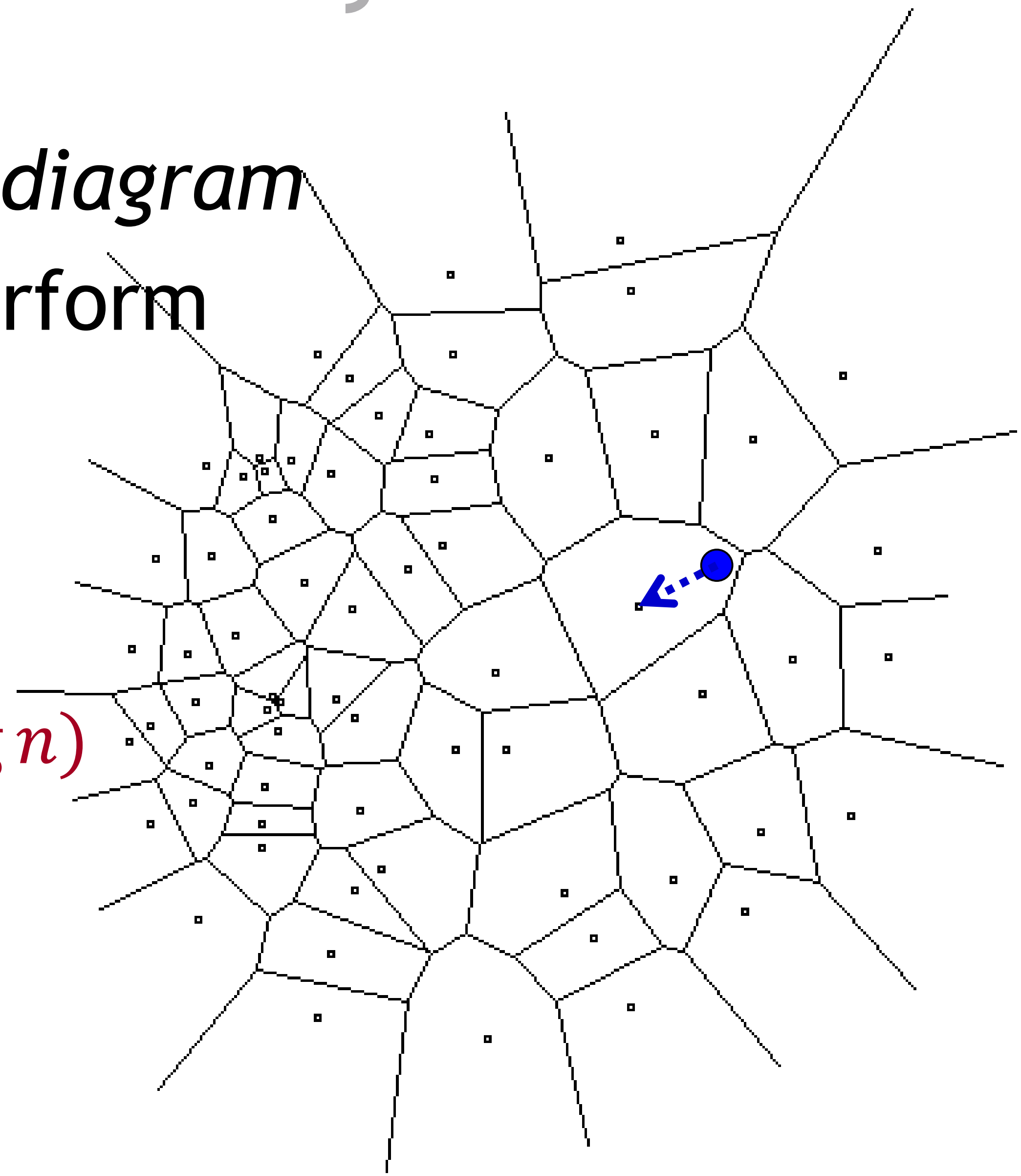
Motivation

- Generic setup:
  - Points model *objects (e.g. images)*
  - Distance models *(dis)similarity measure*
- Application areas:
  - machine learning: k-NN rule
  - speech/image/video/music recognition, vector quantization, bioinformatics, etc…
- Distance can be:
  - Euclidean, Hamming



Low-dimensional: easy

- Compute *Voronoi diagram*
- Given query q, perform *point location*
- Performance:
  - Space: $O(n)$
  - Query time: $O(\log n)$



High-dimensional case

- All exact algorithms degrade rapidly with the dimension d

Algorithm	*Query time*	*Space*
Full indexing	$O(\log n \cdot d)$	$n^{O(d)}$ (Voronoi diagram size)
No indexing – linear scan	$O(n \cdot d)$	$O(n \cdot d)$

Dimension Reduction

- Reduce high dimension?!
  - "flatten" dimension d into dimension $k \ll d$
- Not possible in general: packing bound
- But can if: for a fixed subset of $\Re^d$