CS 4733 Notes: Stanford Arm Inverse Kinematics

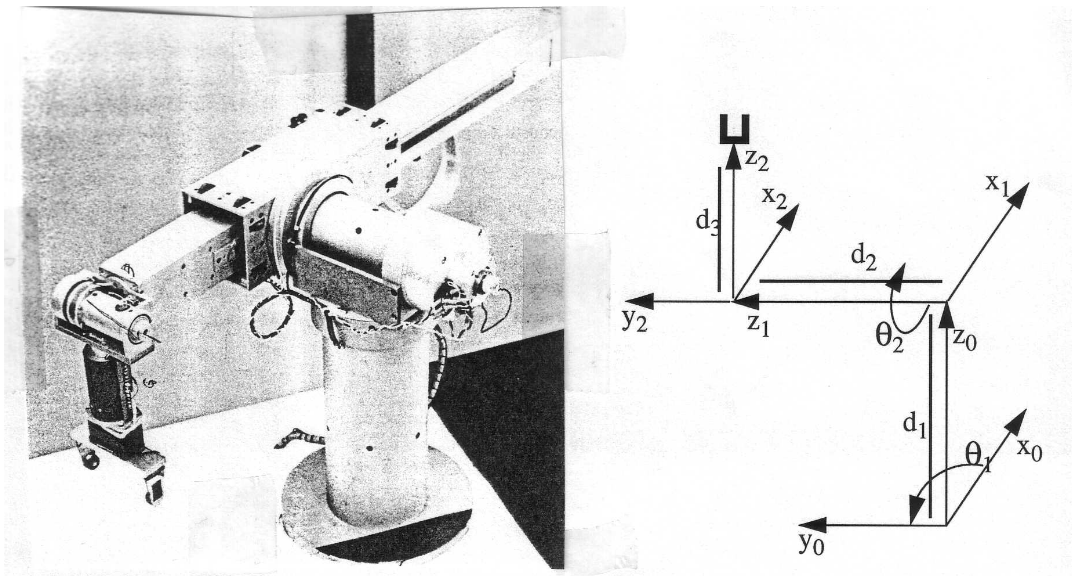

Figure 1: Stanford Robotic Arm. The frame diagram shows the first three joints, which are in a R-R-P configuration (Revolute-Revolute-Prismatic.

1. To solve inverse kinematics for the first 3 joints of the Stanford arm we first look at the matrix T_3^0:

$$T_3^0 = \begin{bmatrix} C_1C_2 & -S_1 & C_1S_2 & C_1S_2d_3 - S_1d_2 \\ S_1C_2 & C_1 & S_1S_2 & S_1S_2d_3 + C_1d_2 \\ -S_2 & 0 & C_2 & C_2d_3 + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} N_x & S_x & A_x & P_x \\ N_y & S_y & A_y & P_y \\ N_z & S_z & A_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   Looking at P (last column of matrix) there are no obvious relationships that isolate a single joint variable (Θ_1,Θ_2,d_3).

2. The technique we employ is to try to rearrange the equations hoping a single variable can be isolated.

$$T_3^0 = T_1^0 T_2^1 T_3^2$$

   Multiplying both sides (on the left) by $(T_1^0)^{-1}$ we get

$$(T_1^0)^{-1}\ T_3^0 \ = \ T_2^1 T_3^2 = T_3^1$$

The matrices are below with only the fourth column filled in:

$$(T_1^0)^{-1}\ T_3^0\ = \left[\begin{array}{cccc} | & | & | & C_1P_x + S_1Py \\ | & | & | & -P_z + d_1 \\ | & | & | & -S_1P_x + C_1Py \\ | & | & | & 1 \end{array}\right] \ = \ \left[\begin{array}{cccc} | & | & | & S_2d_3 \\ | & | & | & -C_2d_3 \\ | & | & | & d_2 \\ | & | & | & 1 \end{array}\right] \ = \ T_3^1$$

We can equate the third entry in last column in each matrix:

$$-S_1P_x + C_1P_y = d_2 \tag{1}$$

We have now isolated Θ_1. To solve for Θ_1, define auxiliary variables r and Φ:

$$r = \sqrt{P_x^2 + P_y^2}\ ;\ \Phi = atan_2(P_y, P_x)$$

Substitute in (1) for P_x,P_y : $P_x = r\ C\ \phi,\ P_y = r\ S\ \phi$

$$-S_1rC\phi + C_1rS\phi = d_2$$

$$r(S\phi C_1 - S_1C\phi) = d_2$$

$$S(\phi - \Theta_1) = \frac{d_2}{r}$$

and by the fact that $C = \pm\sqrt{1 - S^2}$

$$C(\phi - \Theta_1) = \pm\sqrt{1 - (\frac{d_2}{r})^2} = \pm\frac{\sqrt{r^2 - d_2^2}}{r}$$

$$\frac{S(\phi - \Theta_1)}{C(\phi - \Theta_1)} = \frac{\frac{d_2}{r}}{\frac{\pm\sqrt{r^2 - d_2^2}}{r}} = \frac{d_2}{\pm\sqrt{r^2 - d_2^2}}$$

$$\phi - \Theta_1 = atan_2\left(d_2,\ \pm\sqrt{r^2 - d_2^2}\right)$$

Solving for Θ_1, we get

$$\Theta_1 = \phi - atan_2\left(d_2,\ \pm\sqrt{r^2 - d_2^2}\right)$$

$$\Theta_1 = atan_2(P_y,\ P_x) - atan_2\left(d_2,\ \pm\sqrt{r^2 - d_2^2}\right)\ - 2\ solutions$$

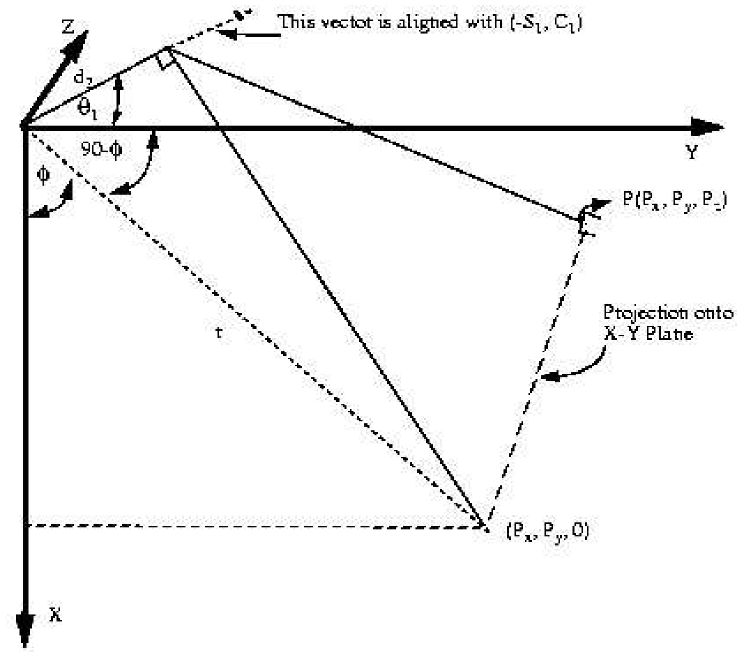

Figure 2: Geometric analysis of Stanford Arm

Geometrically, the figure above describes what the solution for Θ_1 is all about. Project vector (P_x, P_y, P_z) onto the unit vector axis which is along the direction of d_2. The unit vector axis along d_2 is simply the direction of the Y axis after a rotation of Θ_1.

$$\begin{bmatrix} P_x & P_y & P_z \end{bmatrix} \begin{bmatrix} -S_1 \\ C_1 \\ 0 \end{bmatrix} = d_2 \rightarrow -S_1 P_x + C_1 P_y = d_2$$

$$C(\Theta_1 + 90 - \phi) = \frac{d_2}{r} = C(90 - (\phi - \Theta_1)) = S(\phi - \Theta_1)$$

$$If\ S(\phi - \Theta_1) = \frac{d_2}{r}\ then\ C(\phi - \Theta_1) = \pm\sqrt{1 - (\frac{d_2}{r})^2} = \frac{\pm\sqrt{r^2 - d_2^2}}{r}$$

Since we know angle $\phi = atan2(P_y, P_x)$ we can find Θ_1.

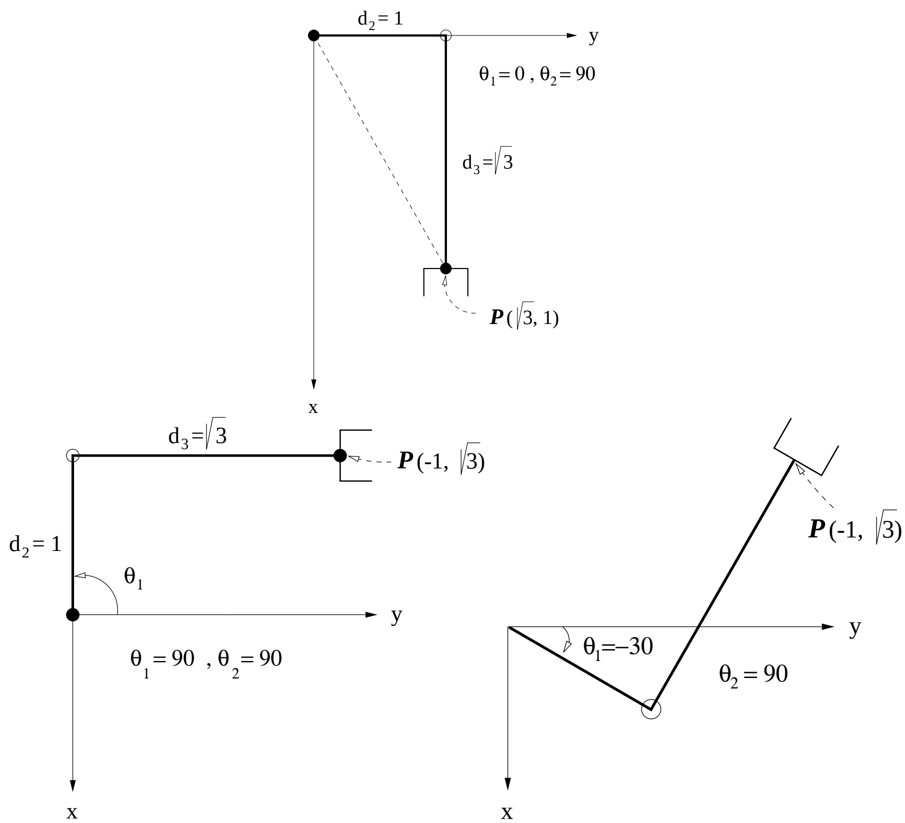

Figure 3: Top: Arm with $\Theta_1 = 0$, $\Theta_2 = 90$. Bottom: multiple solutions for Θ_1 when $P = (-1, \sqrt{3}, 0)$

3. Example: if $\Theta_2 = 90°$, then the manipulator looks as in figure 3(top). (assume $d_3 = \sqrt{3},\ d_2 = 1$).

   Now rotate Θ_1 by 90 and $P = (-1, \sqrt{3}, 0)$ (bottom part of figure 3).

   Using the formula for Θ_1:

$$
\begin{aligned}
\Theta_1 &= atan_2(P_y,\ P_x) - atan_2(d_2,\ \pm\sqrt{r^2 - d_2^2}) \\
&= atan_2(\sqrt{3},\ -1) - atan_2(1,\ \pm\sqrt{3}) \\
\Theta_1 &= 120 - 30 = 90 \\
& \quad or \\
& \quad 120 - 150 = -30
\end{aligned}
$$

   These are the 2 solutions for Θ_1.

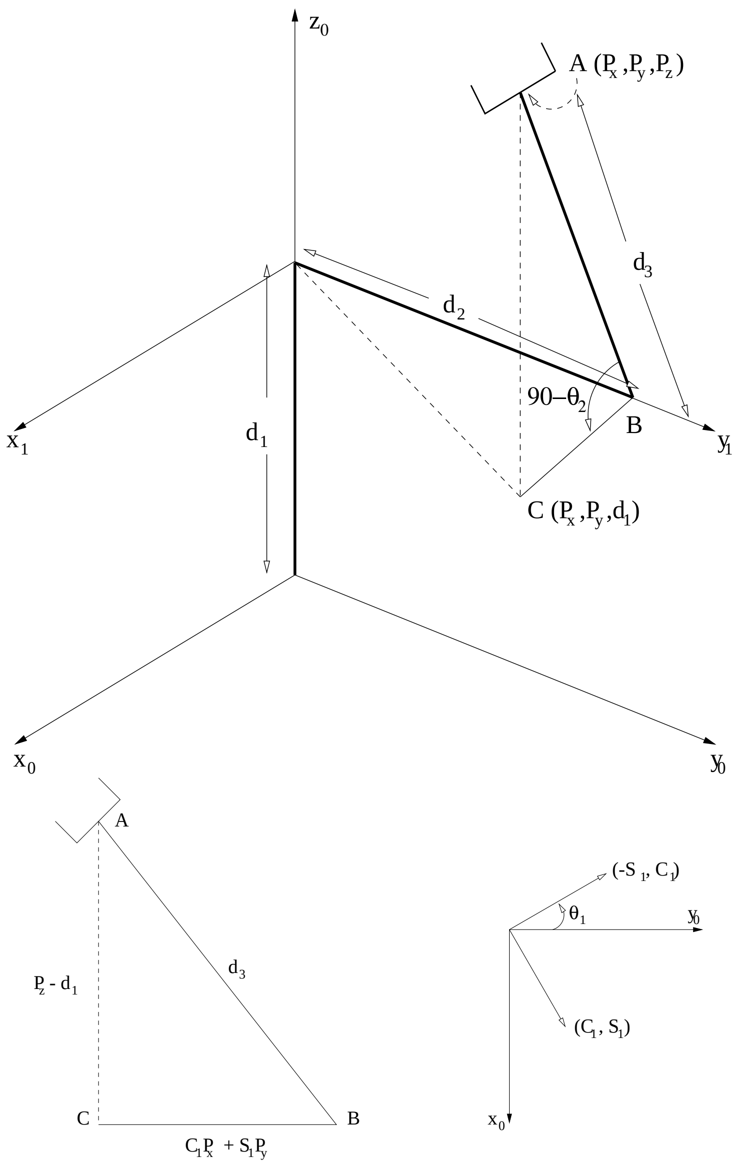

Figure 4: Geometry for finding Θ_2

4. To find Θ_2 from our relationship $(T_1^0)^{-1}\ T_3^0\ =\ T_2^1 T_3^2 = T_3^1$ we notice that:

$$Tan\ \Theta_2 = \frac{S_2 d_3}{C_2 d_3} = \frac{C_1 P_x + S_1 P_y}{P_z - d_1}$$
$$or\ \Theta_2 = atan_2(C_1 P_x + S_1 P_y, P_z - d_1)$$

Figure 4 shows the geometric relationships for finding Θ_2.

As in the previous section, the distance CB in figure 4 is the projection of the vector $(P_x,\ P_y,\ P_z)$ onto the unit vector axis which is along the direction of the X axis rotated by Θ_1.

$$\begin{bmatrix} P_x & P_y & P_z \end{bmatrix} \begin{bmatrix} C_1 \\ S_1 \\ 0 \end{bmatrix} = C_1 P_x + S_1 P_y$$

5. To find d_3

$$(3) \quad C_1 P_x + S_1 P_y = S_2 d_3$$
$$(4) \quad -P_z + d_1 = -C_2 d_3$$

Multiply (3) by S_2 and (4) by C_2 then add the equations:

$$\begin{array}{rlcl} & S_2(C_1 P_x + S_1 P_y) & = & S_2^2 d_3 \\ + & C_2(-Pz + d_1) & = & C^2 d_3 \\ \hline & S_2(C_1 P_x + S_1 P_y) + C_2(P_z - d_1) & = & d_3 \end{array}$$

6. <u>Recapping</u>

$$\begin{array}{rcl} \Theta_1 & = & atan_2(P_x,\ P_y) - atan_2(d_2,\ \pm\sqrt{r^2 - d^2}) \\ \Theta_2 & = & atan_2(C_1 P_x + S_1 P_y,\ P_z - d_1) \\ & & atan_2(-(C_1 P_x + S_1 P_y),\ d_1 - P_z) \\ d_3 & = & S_2(C_1 P_x + S_1 P_y) + C_2(P_z - d_1) \end{array}$$

if $P = (-1, \sqrt{3}, 5)$, and assume $d_1 = 5$ and $d_2 = 1$:

Θ_1	Θ_2	S_1	C_1	S_2	C_2	$S_2(C_1 P_x + S_1 P_y) + C_2(P_z - d_1) = d_3$
90	90	1	0	1	0	$\sqrt{3}$
90	-90	1	0	-1	0	$-\sqrt{3}$
-30	-90	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	-1	0	$\sqrt{3}$
-30	90	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	0	$-\sqrt{3}$

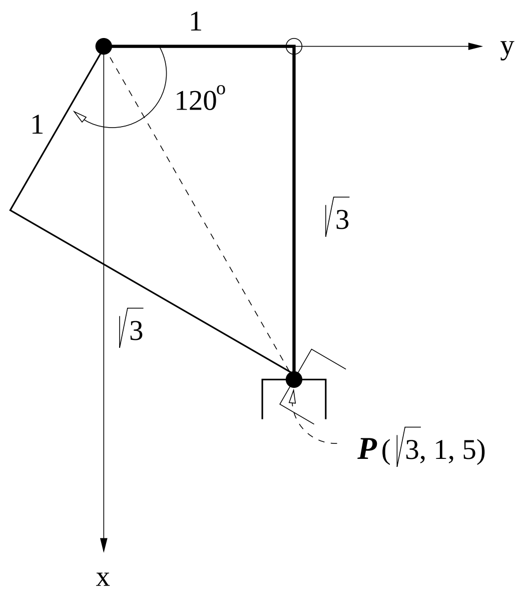

Figure 5: Multiple solutions of arm for $P = (\sqrt{3}, 1, 5)$

7. Find a solution if $P = (\sqrt{3}, 1, 5)$ assume $d_2 = 1$ and $d_1 = 5$.

Θ_1	Θ_2	S_1	C_1	S_2	C_2	$S_2(C_1P_x + S_1P_y) + C_2(P_z - d_1) = d_3$
0	90	0	1	1	0	$\sqrt{3}$
0	-90	0	1	-1	0	$-\sqrt{3}$
-120	-90	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	-1	0	$\sqrt{3}$
-120	90	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	1	0	$-\sqrt{3}$

$$\begin{aligned}\Theta_1 &= atan_2(1, \sqrt{3}) - atan_2(1, \sqrt{3}) = 30 - 30 = 0\\ &= atan_2(1, \sqrt{3}) - atan_2(1, -\sqrt{3}) = 30 - 150 = -120\\ \Theta_2 &= atan_2(\sqrt{3}, 0) = 90^\circ\\ &= atan_2(-\sqrt{3}, 0) = -90^\circ\end{aligned}$$

Geometrically : 4 solutions in all. Figure 5 shows 2 - the other 2 are simply with a negative value of r.

Calculating Angles from Sin, Cos,Tan

- There will be many situations in robotic calculations where the magnitude of the sine, cosine, or tangent of an angle is known, and we need to calculate the magnitude of the angle. Although this seems to be a trivial matter, in reality, it is very important, because there can be grave ambiguities in the answer that may yield an erroneous result, stopping a robot controller from functioning properly. This is true even with a calculator or a computer. To understand this, let's do a simple calculation.

- Suppose that you use your calculator to calculate sin $75°$ as 0.966. If you enter the same number into your calculator and calculate the angle from it, you will find the same $75°$ angle. However, if you do the same with sin $105°$, you will find the same 0.966 as before. Here lies the basic error. The sine of two angles with equal distance from $90°$ is always the same, and thus the calculator always returns the smaller angle. The same is true for cosine and tangent of an angle; the cosine of the plus or minus of the same angle is the same, while the tangent of an angle is the same if $180°$ is added to it.

- To know the exact magnitude of an angle, it is necessary to determine in what quadrant the angle lies. This will enable us to correctly know what the angle really is. However, to determine the quadrant of an angle, it is necessary to know the signs of both the sine and the cosine of the angle.

- In the previous example, if you calculate the values of cos $75°$ and cos $105°$, you will notice that they are respectively 0.259 and -0.259. Considering both the sines and the cosine of $75°$ and $105°$, we can easily calculate their correct values. The same principle is true for the tangent of an angle.

- In robotic calculations, we will encounter the same situation, where tan of angles are generally found. If the simple atan (arctan) function of a calculator or computer is used, it may yield an erroneous result. But if both the sine and the cosine of the angle are found and used in a function, we can calculate the correct angle. Most computer languages have a function called ATAN2(sin, cos), in which the values of the sine and cosine of the angle, entered as arguments, are automatically used to return the value of the angle. In all other situations, either with your calculator or other computer languages, you will have to write such a function. As a result, it is generally necessary to find two equations for each angle, one that yields the sine of the angle, and one that yields the cosine of the angle. Based on the sign of the two, we will determine the quadrant and, thus, the correct value of the angle.

- The following is a summary of rules for calculating the angles in each quadrant (you may program this into your robotic routines or your calculator for future use):

  - If sin is positive and cos is positive, the angle is in quadrant 1, then angle = atan(a).

  - If sin is positive and cos is negative, the angle is in quadrant 2, then angle = 180 - atan(a).

  - If sin is negative and cos is negative, the angle is in quadrant 3, then angle = 180 + atan(a).

  - If sin is negative and cos is positive, the angle is in quadrant 4, then angle = -atan(a).

- The program should check to see if either the sine or the cosine are zero. In that case, instead of calculating the tangent, it should use the cosine or the sine to calculate the angle to prevent an error.

- Useful formulae: Law of Cosines: $C^2 \ = \ A^2 \ + \ B^2 \ - \ 2\,A\,B\,Cos(\theta)$

  Double Angle Formulas:
  $Sin(\theta_1 \pm \theta_2) \ = \ Sin\theta_1 Cos\theta_2 \pm Sin\theta_2 Cos\theta_1$
  $Cos(\theta_1 \pm \theta_2) \ = \ Cos\theta_1 Cos\theta_2 \mp Sin\theta_1 Sin\theta_2$