CS 4733 Class Notes: Composite Rotations

1 Euler Angles

We can represent rotation in space by rotating about the 3 coordinate axes, X, Y, Z. The particular order we do this is somewhat arbitrary. If we choose 1 of the 3 axes as the first rotation, then there are 2 choices for the second rotation axis (we can't use the same axis twice in a row), and 2 choices for the third axis. This leaves $3 \times 2 \times 2 = 12$ choices for possible sequences of 3 rotations. The angles for each of the 3 rotations are referred to as Euler angles.

A very popular and widely used set of Euler angles is $Z\,Y\,Z$: a rotation about the world Z frame, followed by a rotation about Y in the new local frame, followed by another rotation about Z in the local frame.

$$Rot(Z,\phi)\,Rot(Y,\theta)\,Rot(Z,\psi) \;=\; \begin{bmatrix} C\phi C\theta C\psi - S\phi S\psi & -C\phi C\theta S\psi - S\phi C\psi & C\phi S\theta & 0 \\ S\phi C\theta C\psi + C\phi S\psi & -S\phi C\theta S\psi + C\phi C\psi & S\phi S\theta & 0 \\ -S\theta C\psi & S\theta S\psi & C\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

We can derive this equation by simpling multiplying the three elementary rotation matrices together.

Once we know the form of this transform matrix, we can use it to find the Euler angles given the transformation matrix. Said another way, if we have a final position and orientation matrix, we can break out of this matrix the Euler anlges that can get us there. So given an arbitrary transform:

$$\begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

$$\phi \;=\; ATAN2(a_y\,,\,a_x)\;\, or\;\, \phi = ATAN2(-a_y\,,\,-a_x)$$

$$\psi \;=\; ATAN2(-n_x S\phi \,+\, n_y C\phi\,,\, -o_x S\phi \,+\, o_y C\phi)$$

$$\theta \;=\; ATAN2(a_x C\phi \,+\, a_y S\phi\,,\, a_z)$$

2 Roll, Pitch, Yaw Angles

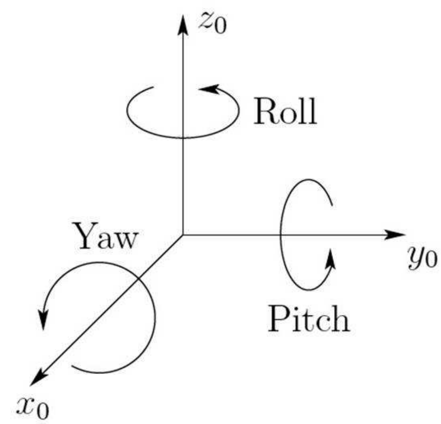

Figure 2.11: Roll, pitch, and yaw angles.

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Figure 1: Roll, Pitch and Yaw angles

A common representation for orientation is to use roll, pitch and yaw angles. They are defined as rotations about the world or global coordinate frame. The order of rotation is X, Y, Z, which in global coordinate frames we can write the composite rotation as:

$$R = Rot(Z, \phi)\, Rot(Y, \theta)\, Rot(X, \psi)$$

$$Rot(Z, \phi)\, Rot(Y, \theta)\, Rot(X, \psi) = \begin{bmatrix} C\phi C\theta & -S\phi C\psi + C\phi\, S\theta\, S\psi & S\phi\, S\psi + C\phi\, S\theta\, C\psi & 0 \\ S\phi C\theta & C\phi\, C\psi + S\phi S\theta S\psi & -C\phi\, S\psi + S\phi\, S\theta\, C\psi & 0 \\ -S\theta & C\theta\, S\psi & C\theta\, C\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

Solving the inverse problem is similar to Euler angles. You need to isolate components from the matrix, solve for one of the angles, and use these solutions to find the others. You can think of this representation intuitively as if you are in a boat, headed along the positive Z axis. As you move through the water, you pitch about the Y axis (your bow goes up and down), you yaw about the X axis (move from side to side) and roll about the Z axis (roll left and right as you proceed),

3 Equivalent Axis Transform

The rotation matrix for a rotation by angle ϕ about any unit vector axis $U = (\mu_1, \mu_2, \mu_3)$ is:

$$R(\phi,\mu) = \begin{bmatrix} \mu_1^2 V\phi + C\phi & \mu_1\mu_2 V\phi - \mu_3 S\phi & \mu_1\mu_3 V\phi + \mu_2 S\phi \\ \mu_1\mu_2 V\phi + \mu_3 S\phi & \mu_2^2 V\phi + C\phi & \mu_2\mu_3 V\phi - \mu_1 S\phi \\ \mu_1\mu_3 V\phi - \mu_2 S\phi & \mu_2\mu_3 V\phi + \mu_1 S\phi & \mu_3^2 V\phi + C\phi \end{bmatrix} \qquad Note: V\phi = 1 - C\phi$$

We can express this arbitrary rotation as a combination of known canonical rotations about the X, Y, and Z axes. The trick is to rotate the axis of rotation U so that it coincides with one of the known rotation axes, perform the rotation by angle ϕ about this axis, and then reverse the set of rotations that brought the axis U to coincide with a known axis.

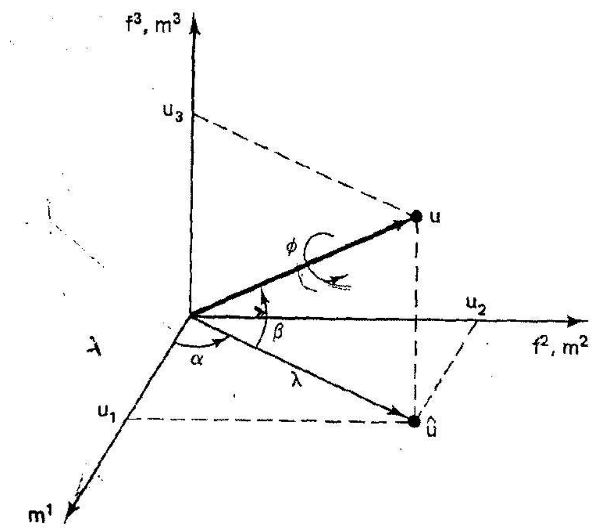

Figure 2: Axes used to establish the equivalent axis transform

One way to do this (see figure 3) is to perform the following operations:

1. Rotate the axis U about the global Z axis to bring it onto the $X - Z$ plane.
2. Rotate about the global Y axis to make U coincident with the global X axis.
3. Since U is now coincident with X, we can perform a rotation about X of ϕ.
4. Now, we reverse the rotations that brought U coincident with the global X axis, and the result is the initial frame rotated about axis U by angle ϕ

We can define the rotation about arbitrary axis U to be a combination of **KNOWN** rotations about X, Y, and Z axes:

$$R(\phi,\mu) = R_3(\alpha)R_2(-\beta)R_1(\phi)R_2(\beta)R_3(-\alpha)$$

$$R(\phi,\mu) = \begin{bmatrix} C\alpha & -S\alpha & 0 \\ S\alpha & C\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C(-\beta) & 0 & S(-\beta) \\ 0 & 1 & 0 \\ -S(-\beta) & 0 & C(-\beta) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\phi & -S\phi \\ 0 & S\phi & C\phi \end{bmatrix} \begin{bmatrix} C\beta & 0 & S\beta \\ 0 & 1 & 0 \\ -S\beta & 0 & C\beta \end{bmatrix} \begin{bmatrix} C(-\alpha) & -S(-\alpha) & 0 \\ S(-\alpha) & C(-\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Although the angles α, β appear in the matrices, we can see from the figure that:

$$C\alpha = \frac{\mu_1}{(\mu_1^2 + \mu_2^2)^{1/2}} \quad ; \quad S\alpha = \frac{\mu_2}{(\mu_1^2 + \mu_2^2)^{1/2}} \quad ; \quad C\beta = (\mu_1^2 + \mu_2^2)^{1/2} \quad ; \quad S\beta = \mu_3$$

Going the other way, if we are given any rotation matrix, we can find its axis and amount of rotation. This is done by observing that if we add the elements of the diagonal of the matrix $R(\phi, \mu)$ (known as the *Trace* of the matrix) we get:

$$Trace(R) \;=\; (\mu_1^2 + \mu_2^2 + \mu_3^2)V\phi + 3C\phi \;=\; V\phi + 3C\phi \;=\; 1 + 2C\phi$$

We can then solve for ϕ. Once we know ϕ, we can then go back into the matri x R and find the components of U:

$$\phi = ArcCos\left[\frac{Trace(R) - 1}{2}\right] \quad ; \quad U \;=\; \frac{1}{2S\phi}\begin{bmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{bmatrix}$$

Example: Given the matrix:

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

We can solve for the axis and rotation angle.

$$\phi \;=\; ArcCos(-\frac{1}{2}) \;=\; \pm 120^\circ$$

$$\phi \;=\; 120^\circ \,,\, U \;=\; \frac{1}{\sqrt(3)}\begin{bmatrix} 1-0 \\ 1-0 \\ 1-0 \end{bmatrix} \;\; ; \;\; or \;\; \phi \;=\; -120^\circ \,,\, U \;=\; -\frac{1}{\sqrt(3)}\begin{bmatrix} 1-0 \\ 1-0 \\ 1-0 \end{bmatrix} \tag{4}$$

Note: no matter how many rotations about the X,Y or Z axes we do, we can chain them into a SINGLE rotation matrix by multiplying them all together. And using the euqivalent axis idea, all those composite rotations are equivalent to a SINGLE rotation about a specifeid axis in space.