CS W4733 CLASS NOTES

SPECIFIYING POSITION AND ORIENTATION

- We need to describe in a compact way the position of the robot. Think of this as specifying where the gripper of a robot is located and what direction it is pointing in.

- In 2 dimensions (planar robot), there are 3 degrees of free-dom (DOF): **X , Y** position and 1 orientation parameter θ

- In 3 dimensions there are 6 DOF: **X , Y, Z** position and 3 angular orientation parameters specifying orientation of the gripper in space. There are a number of ways to specify these angles wich we will discuss.

- In 3-D, we can specify both position and orientation using a translation vector (3x1 vector) and a rotation matrix (3x3) which encodes the orientation information.

ROTATION MATRIX

- Orthonormal matrix: columns are orthogonal basis vectors of unit length.
- Row vectors are also orthogonal unit vectors
- Determinant = 1 (Right handed system) -1(Left handed)
- Columns establish axes of new coordinate system with respect to previous frame
- Transpose of rotation matrix is its inverse
- Eigenvectors of the matrix form the axis of rotation.

$$\mathbf{ROT(X,}\theta\mathbf{)} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \cos\theta & -\sin\theta \\ \mathbf{0} & \sin\theta & \cos\theta \end{bmatrix} \quad \mathbf{ROT(Y,}\theta\mathbf{)} = \begin{bmatrix} \cos\theta & \mathbf{0} & \sin\theta \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ -\sin\theta & \mathbf{0} & \cos\theta \end{bmatrix}$$

$$\mathbf{ROT(Z,}\theta\mathbf{)} = \begin{bmatrix} \cos\theta & -\sin\theta & \mathbf{0} \\ \sin\theta & \cos\theta & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

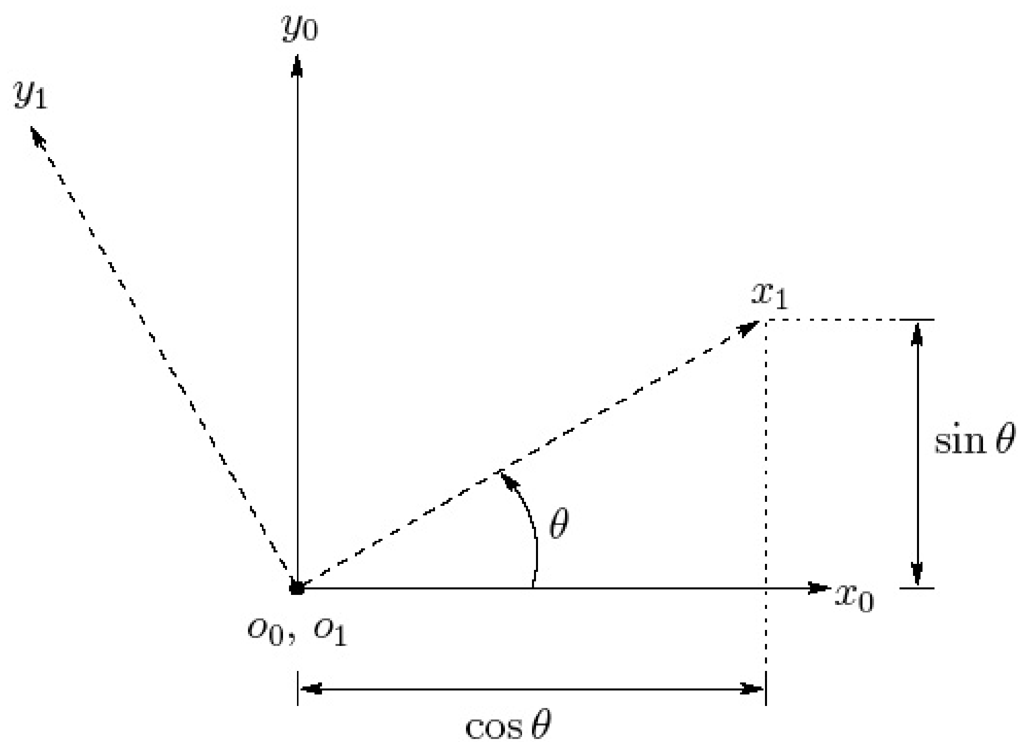

Figure 2.2: Coordinate frame $o_1x_1y_1$ is oriented at an angle θ with respect to $o_0x_0y_0$.

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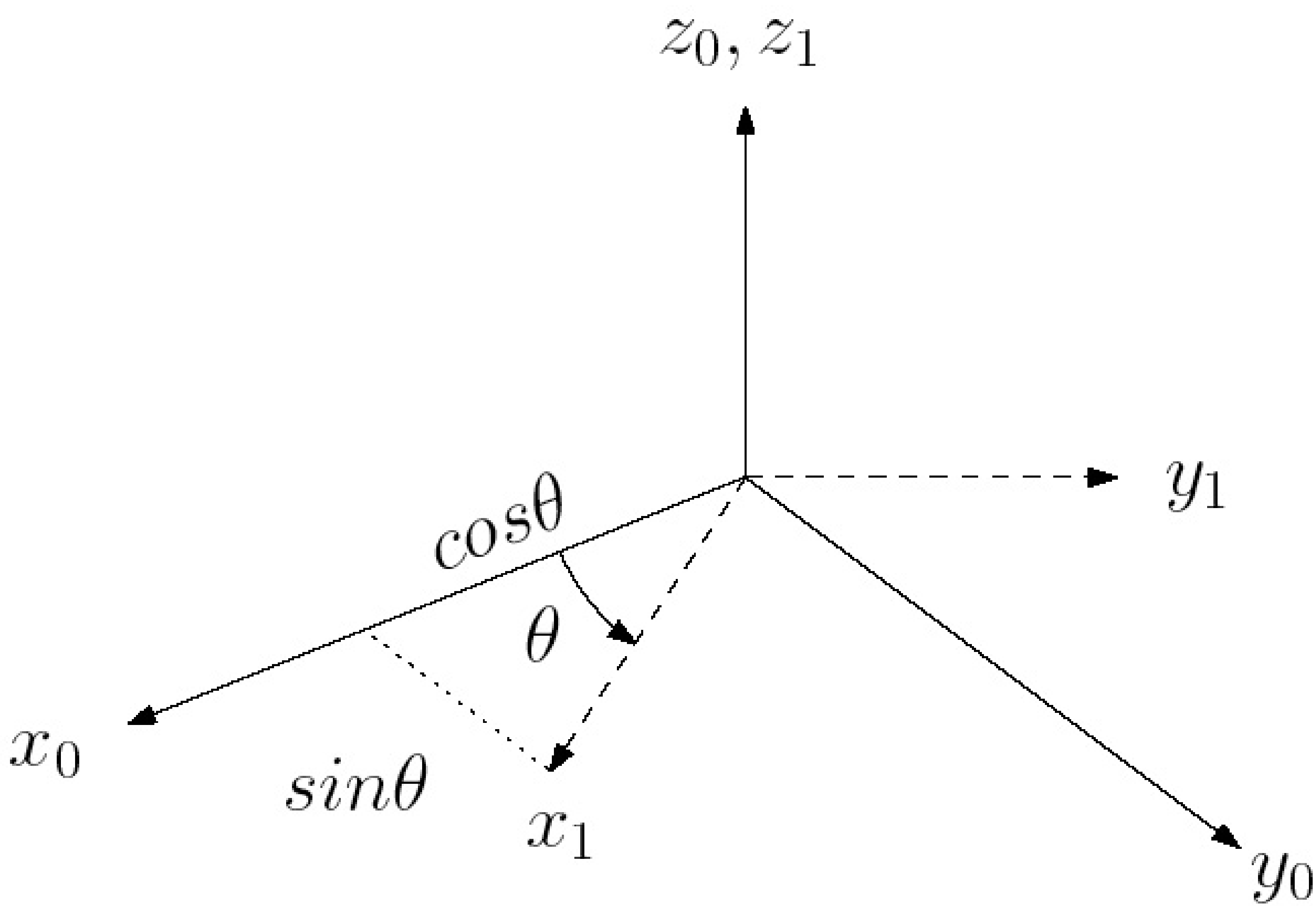


Figure 2.3: Rotation about z_0 by an angle θ.

EXAMPLE:

$$\mathbf{ROT(Z,90)} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If we transform the x axis by this transform we get:

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Order is important: look at the composite rotations:

1. **ROT(Z,90) ROT(Y,90)**

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

2. **ROT(Y,90) ROT(Z,90)**:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Going left to right, rotations are done in the new or local frame established by the previous rotations. As we go right to left, the transformations are done in global coordianates.

HOMOGENEOUS COORDINATES

- Transforming a 3-D point with a 3x3 matrix allows for scaling, shearing and rotation, but not translation.

- By using a 4x4 matrix, we can add translation to the transformation. Since we need to apply 4x4 matrices to 4-D vectors, we add an arbitrary scaling factor (typically with value 1) to the 3-D coordinates of a point. You can think of the 3-D point as the projection into 3-D of a 4-D point.

- Homogeneous coordinates allow us to embed a lower dimensional space in a higher dimensional space. An example is a homogeneous vector:

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \\ \mathbf{w} \end{bmatrix}$$

which is a 4 space vector that can be projected into 3 space as

$$\begin{bmatrix} \dfrac{\mathbf{x}}{\mathbf{w}} \\ \dfrac{\mathbf{y}}{\mathbf{w}} \\ \dfrac{\mathbf{z}}{\mathbf{w}} \end{bmatrix}$$

The 4x4 homogeneous transform contains a 3x3 rotation matrix and a 3x1 translation vector. Here is a 4x4 translation transform:

$$\begin{bmatrix} 1 & 0 & 0 & p_x \\ 0 & 1 & 0 & p_y \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x+p_x \\ y+p_y \\ z+p_z \\ 1 \end{bmatrix}$$

Here are the 4x4 rotation matrices:

$$\mathbf{ROT(X,\theta)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{ROT(Y,\theta)} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{ROT(Z,\theta)} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Transforms contain BOTH rotation and translation information. Its important to remember that translation is done first:

$$\begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & p_x \\ 0 & 1 & 0 & p_y \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A 4 x 4 Homogeneous transform T has an inverse T^{-1} which is found by taking the transpose of the 3x3 rotation matrix in the transform, and forming a new 4th column vector which is the negative dot product of the original 4th column (the translation vector) with each of the other 3 column vectors of the transform:

$$T = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} n_x & n_y & n_z & -p\cdot n \\ o_x & o_y & o_z & -p\cdot o \\ a_x & a_y & a_z & -p\cdot a \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Multiplying the inverse times the matrix gives us the identity:

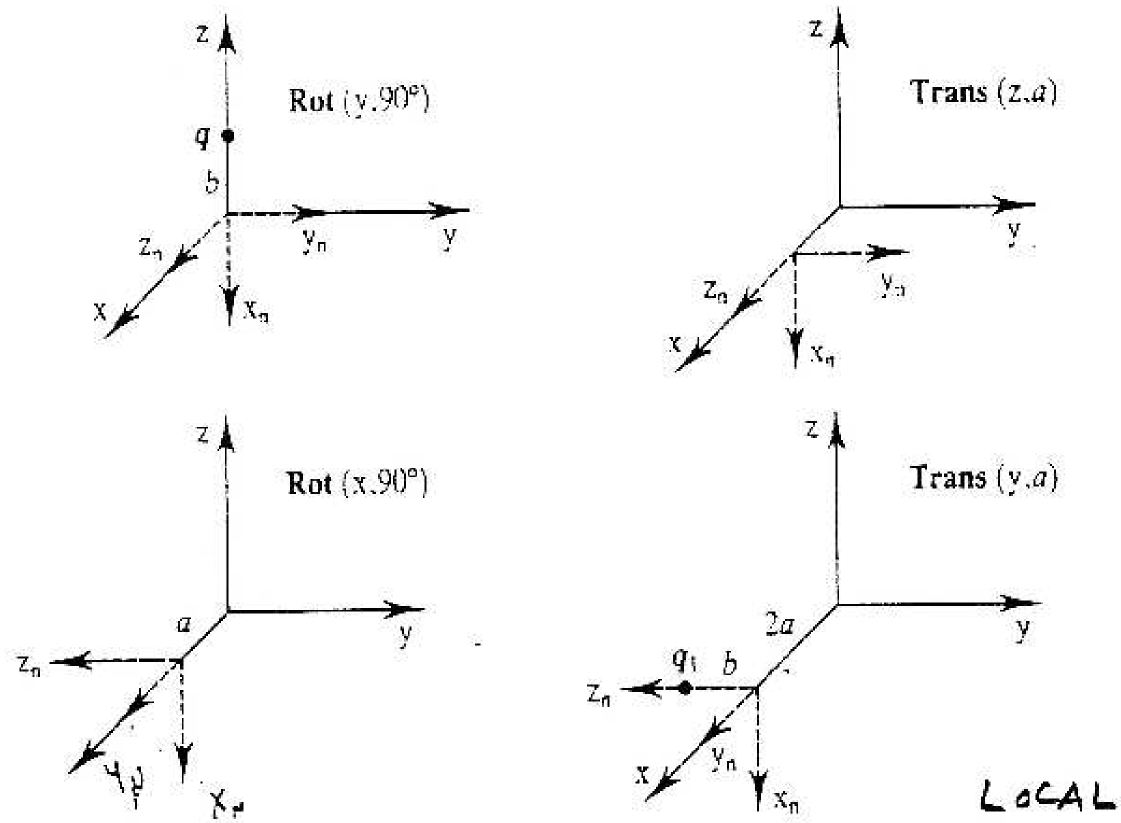
$$T^{-1}\ T = \begin{bmatrix} n_x & n_y & n_z & -p\cdot n \\ o_x & o_y & o_z & -p\cdot o \\ a_x & a_y & a_z & -p\cdot a \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} n\cdot n & n\cdot o & n\cdot a & -p\cdot n+p\cdot n \\ o\cdot n & o\cdot o & o\cdot a & -p\cdot o+p\cdot o \\ a\cdot n & a\cdot o & a\cdot a & -p\cdot a+p\cdot a \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

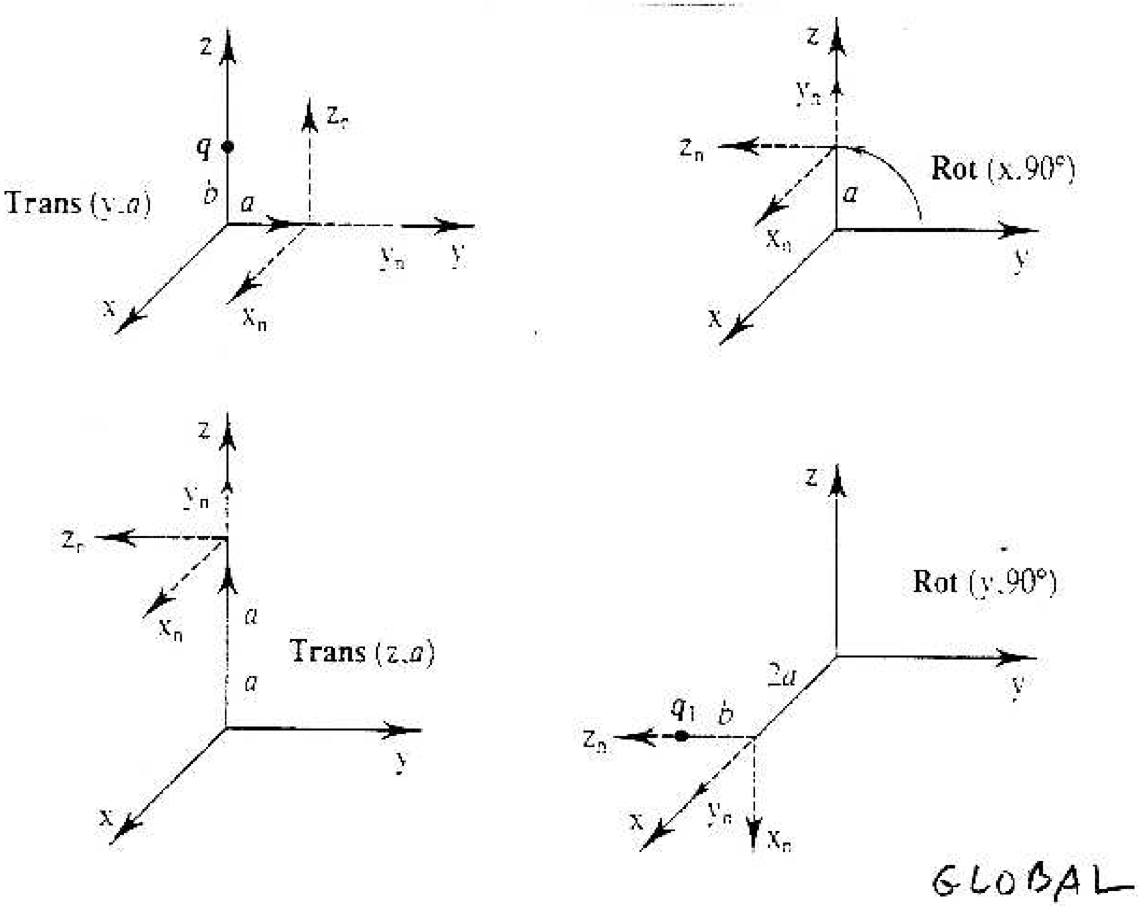
This shows $\mathbf{T}^{-1}$ is left inverse of **T**. But we can also show that if a left inverse and right inverse exist, they must be the same. Given **L** is a left inverse of **T** and **R** is a right inverse of **T**, by definition of inverse we have:

$$L\,T = I\ ,\ \ T\,R = I$$

$$L = L\,I = L\,(T\,R) = (L\,T)\,R = I\,R = R$$

ROT(Y,90) TRANS(Z,a) ROT(X,90) TRANS(Y,a)





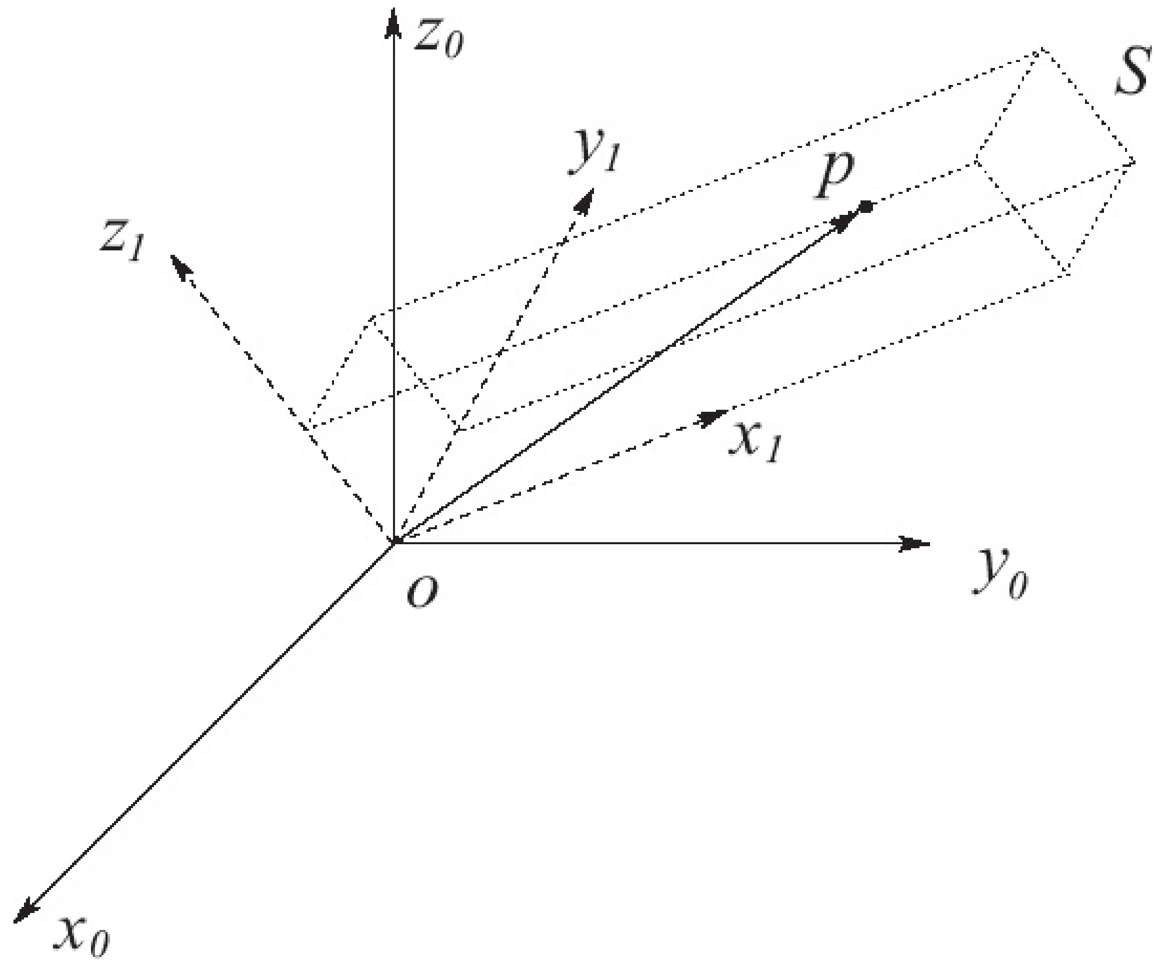


Figure 2.5: Coordinate frame attached to a rigid body.

Three Interpretations of a Transform

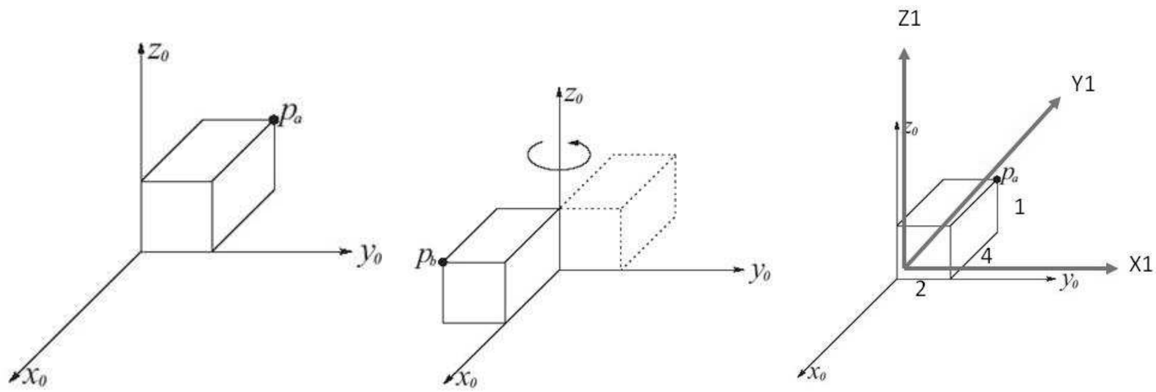


Figure 1: Left: Base frame (X_0, Y_0, Z_0) with box object and point P_a. Middle: Transformed point P_b after rotation by 180 degrees. Left: Box frame(X_1, Y_1, Z_1), with box dimensions.

- In the first interpretation of a transform, we note that a transform can simply represent the rotation and translaton of a known point in the same frame. Given the point P_a, whose coordinates in frame 0 are:

$$P_a^0 \;=\; \begin{bmatrix} -4 \\ 2 \\ 1 \\ 1 \end{bmatrix} \;;\; ROT(Z,180) \;=\; \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \;;\; P_b^0 \;=\; ROT(Z,180)\; P_a^0 \;=\; \begin{bmatrix} 4 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

- In the second interpretation of a transform, we can use a transform to express the coordinates of a point in one frame in the other frame:

$$P_a^1 \;=\; \begin{bmatrix} 2 \\ 4 \\ 1 \\ 1 \end{bmatrix} \;;\; P_a^0 \;=\; T_1^0\; P_a^1 \;=\; = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 1 \\ 1 \end{bmatrix} \;=\; \begin{bmatrix} -4 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

- Finally, we can simply say that T_1^0 represents a new frame where the new X, Y, Z basis vectors are the first 3 columns of T, and the new origin is the 4th column.

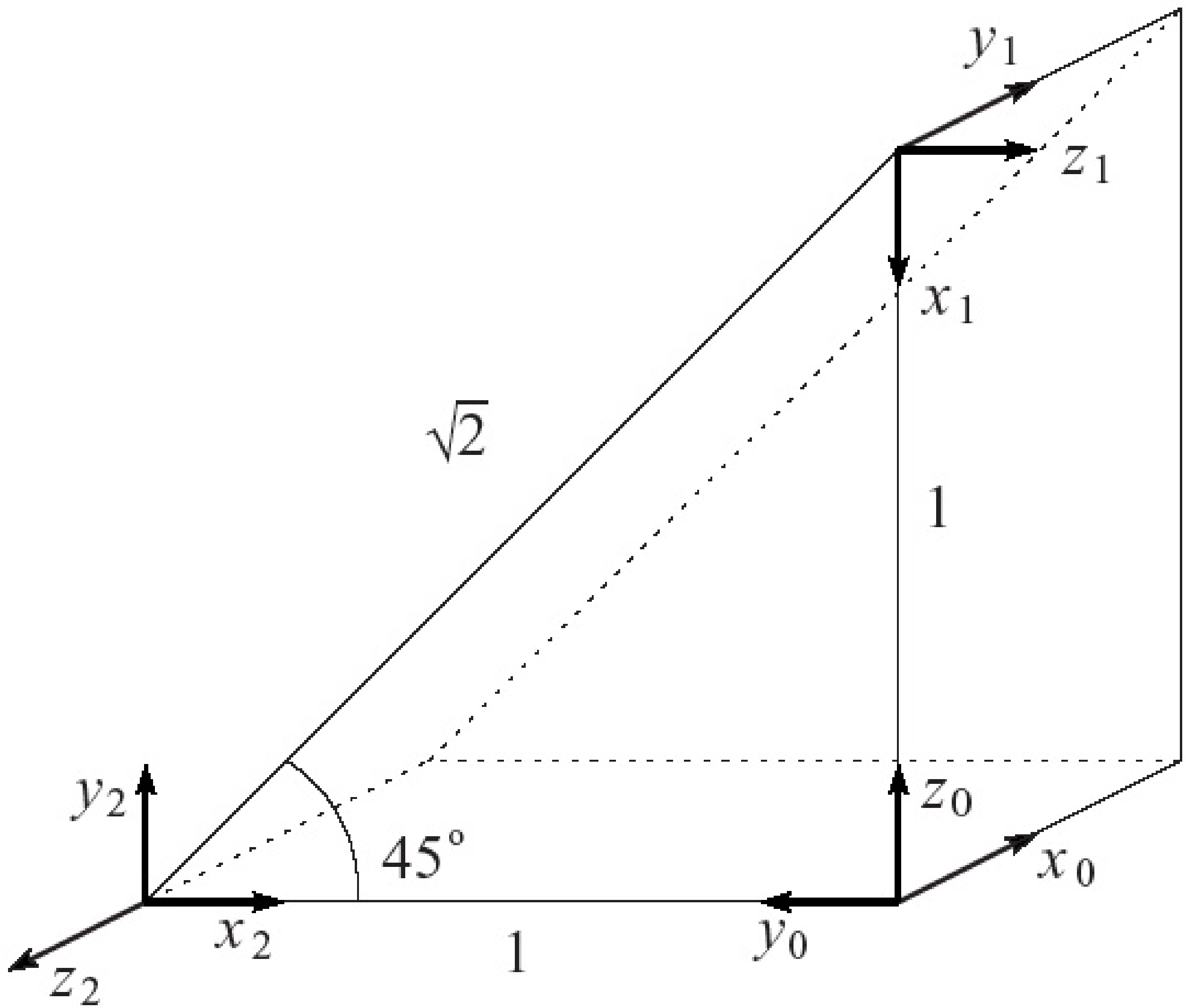


Figure 2.13: Diagram for Problem 2-38.

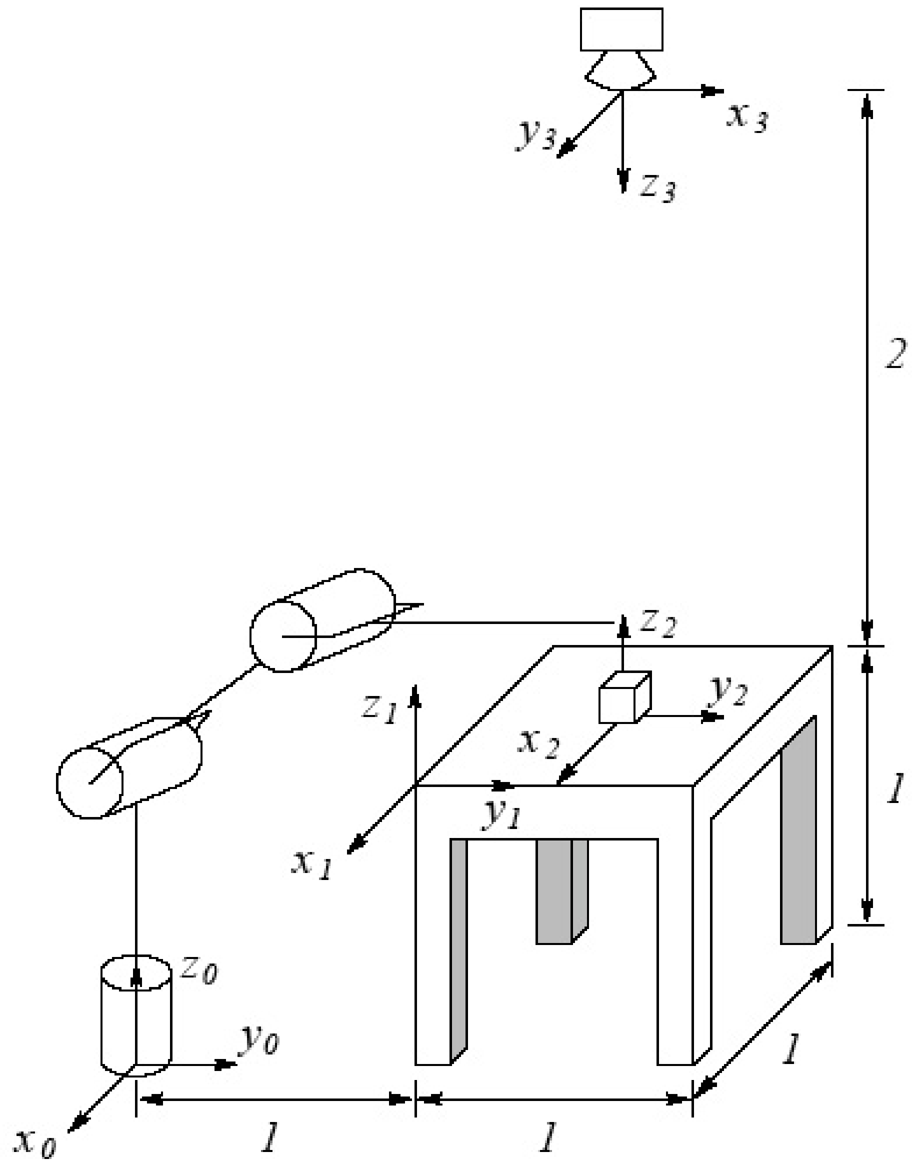
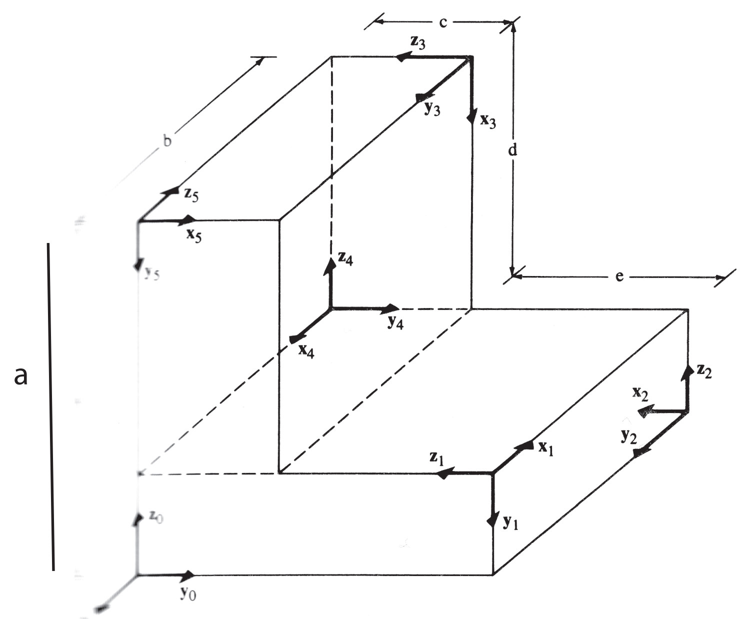


Figure 2.14: Diagram for Problem 2-39.



c
b
d
e
a
z_3
y_3
x_3
z_5
x_5
y_5
z_4
y_4
x_4
z_2
x_2
y_2
z_1
x_1
y_1
z_0
y_0