

An Overview of some Basic Properties of Sponsored Search Auctions

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1 Single Keyword, Multiple Slots

There are m positions to be allocated among n bidders, where $n > m$. We assume that the (expected) click-through rate of bidder i in position j is of the form $\alpha_i \gamma_j$, i.e. separable into an advertiser effect $\alpha_i \in [0, 1]$ and position effect $\gamma_j \in [0, 1]$. We assume that $\gamma_1 > \gamma_2 > \dots > \gamma_k > 0$ and let $\gamma_j = 0$ for $j > k$. We will sometimes refer to α_i as the *relevance* of bidder i . It is useful to interpret γ_j as the probability that an ad in position j will be noticed, and α_i as the probability that it will be clicked on if noticed.

Bidder i has value v_i for each click. Bidders have quasi-linear utility, so that the utility to bidder i of obtaining position j at a price of p per click is

$$\alpha_i \gamma_j (v_i - p).$$

The auctioneer observes the advertiser effects, but the bidders' values remain private.

A weight w_i is associated with agent i , and agents bid for position. If agent i bids b_i , his corresponding *reported score*, or simply his *score*, is $s_i = w_i b_i$. His *true score* is $r_i = w_i v_i$. Agents are ranked by score, so that the agent with highest score is ranked first, and so on. Note that the weights may depend on the advertiser effects, but not on the bidder values, because the latter remain unobservable. We also disallow weights that depend on the agent bids.

Throughout, agents are numbered such that agent i obtains position i , unless mentioned otherwise. An agent pays per click the lowest bid necessary to retain his position, so that the agent in position j pays $\frac{w_{j+1}}{w_j} b_{j+1}$. We refer to this payment rule as “second pricing.”

2 Incentives

2.1 Dominant Strategies

The second-price payment rule is reminiscent of the second-price (Vickrey) auction used for selling a single item. Recall that in a Vickrey auction it is a dominant strategy for a bidder to reveal his true value for the item [18]. However, using a second-price rule in a position auction does not yield an incentive-compatible mechanism, either in dominant strategies or *ex post* Nash equilibrium.¹

¹Unless of course there is only a single position available, since this is the single-item case. With a single position and a second-price payment rule, a position auction is dominant-strategy incentive-compatible for any weighting scheme.

With a second-price rule there is no incentive for a bidder to bid higher than his true value per click: this either leads to no change in the outcome, or a situation in which he will have to pay more than his value per click for each click received, resulting in a negative payoff. However, there may be an incentive to shade true values with second pricing.

Claim 1 *With second pricing and $m \geq 2$, truthful bidding is not a dominant strategy nor an ex post Nash equilibrium in a position auction.*

Example. There are two agents and two positions. The advertiser effects are $\alpha_1 = \alpha_2 = 1$ whereas the position effects are $\gamma_1 = 1$ and $\gamma_2 = 1/2$. Agent 1 has a value of $v_1 = 6/w_1$ per click, and agent 2 has a value of $v_2 = 4/w_2$ per click (recall that weights cannot depend on bidder values). Suppose agent 2 bids truthfully. If agent 1 also bids truthfully, he wins the first position and obtains a payoff of $6/w_1 - 4/w_1 = 2/w_1$. However, if he shades his bid below $4/w_1$, he obtains the second position at a cost of 0 per click yielding a payoff of $\frac{1}{2}(6/w_1 - 0) = 3/w_1$. Hence truthful bidding is not a dominant strategy, and neither is it an *ex post* Nash equilibrium.

Given that second pricing is not strategy-proof, we may ask whether there exists a payment rule that, together with a given weighting scheme, makes it a dominant strategy for the agents to bid their true values. To answer this question, we need to temporarily augment the notation just introduced to cast the problem in the framework of mechanism design. Agent i 's value is his *type*, and we denote it by t_i rather than v_i . Each agent has a value function parametrized by its type, which gives the utility derived from a position before any payments are issued:

$$v_i(j; t_i) = \alpha_i \gamma_j t_i.$$

Agent i 's utility for position j at a total price of q (price per click times clickthrough rate), now parametrized by type, is again quasi-linear:

$$u_i(j, p; t_i) = v_i(j; t_i) - q.$$

Let $z_i(b)$ be the position allocated to i when the vector of bids is $b = (b_i)_{i \in N}$, and similarly let $p_i(b)$ be agent i 's total payment.

Holmstrom's Lemma [7] gives a necessary condition on the structure of the payment rule p if we wish to implement allocation rule z in dominant strategies. Under the restriction that a bidder with value 0 per click does not pay anything, the lemma states that there is a unique candidate payment rule that achieves dominant-strategy incentive compatibility for a given allocation rule. Let $V_i(t_i, b_{-i})$ be agent i 's full information maximum value when the others are bidding b_{-i} and his own value per click is t_i :

$$V_i(t_i, b_{-i}) = \max_{b_i \geq 0} u_i(z_i(b), p_i(b); t_i).$$

The following version of Holmstrom's lemma is provided in Milgrom [13], who shows that it is a consequence of a variant of the envelope theorem [14].

Lemma 1 *Suppose v_i is continuously differentiable in type. If truthful reporting is a dominant strategy for agent i , his payment must satisfy*

$$p_i(b_i, b_{-i}) = -V_i(0, b_{-i}) + v_i(z(b_i, b_{-i}); b_i) - \int_0^{b_i} \frac{\partial v_i}{\partial t_i}(z(\tau, b_{-i}), \tau) d\tau. \quad (1)$$

To apply this lemma, we split the interval of integration into $I_j = \left[\frac{w_{j+1}b_{j+1}}{w_i}, \frac{w_jb_j}{w_i} \right]$ for $i \leq j \leq N$. For $\tau \in I_j$, we have

$$\frac{\partial v_i}{\partial t_i}(z(\tau, b_{-i}), \tau) = \alpha_i \gamma_j,$$

and hence

$$\int_{I_j} \frac{\partial v_i}{\partial t_i}(z(\tau, b_{-i}), \tau) d\tau = \alpha_i \gamma_j \left(\frac{w_j b_j}{w_i} - \frac{w_{j+1} b_{j+1}}{w_i} \right)$$

Noting that $V_i(0, b_{-i}) = 0$ and $v_i(z(b_i, b_{-i}), b_i) = \alpha_i \gamma_i b_i$, equation (1) then evaluates to

$$\begin{aligned} p_i(b_i, b_{-i}) &= \alpha_i \gamma_i b_i - \sum_{j=i}^{n-1} \alpha_i \gamma_j \left(\frac{w_j b_j}{w_i} - \frac{w_{j+1} b_{j+1}}{w_i} \right) - \alpha_i \gamma_n \left(\frac{w_n b_n}{w_i} \right) \\ &= \sum_{j=i+1}^n (\alpha_i \gamma_{j-1} - \alpha_i \gamma_j) \frac{w_j b_j}{w_i} \end{aligned} \quad (2)$$

This is the total payment that should be charged. To reduce this to a payment per click, simply divide by $\alpha_i \gamma_i$. When the allocation rule z and the utility $V_i(0, b_{-i})$ to the lowest possible type are fixed as in our scenario, Lemma 1 uniquely determines the payment rule that could yield truthfulness. In this case, the payment rule does indeed align each agent's incentives with the auctioneer's objective, which gives us strategyproofness; see Aggarwal et al. [1].

Lahaie [10] applied Holmstrom's lemma to derive the strategyproof payment rules for models of Yahoo and Google's position auctions, which coincide with the special cases where $w_i = 1$ and $w_i = \alpha_i$, respectively. Iyengar and Kumar [8] independently derived these rules using similar techniques. Aggarwal et al. [1] derived payment rule (2) and confirmed uniqueness from first principles, which gives some economic insight into why the rule works.

Using payment rule (2), the auctioneer is aware of the true scores of the bidders (since they reveal their values truthfully), and hence ranks them according to their true scores. We show in Section 3 that ranking bidders in decreasing order of $\alpha_i v_i$ is in fact efficient. Since the VCG mechanism is the unique mechanism that is efficient, truthful, and ensures bidders with value 0 pay nothing (by the Green-Laffont theorem [5]), the weighting scheme $w_i = \alpha_i$ and payment scheme (2) constitute exactly the VCG mechanism.

In the VCG mechanism an agent pays the externality he imposes on others. To understand payment (2) in this sense, note that when $w_i = \alpha_i$, agent j 's payment is the added utility (due to an increased clickthrough rate) that agents in positions $j+1$ and lower would receive if they were all to move up a position.

2.2 Nash Equilibrium

In typical position auctions such as those run by Yahoo and Google, bidders can adjust their bids up or down at any time. As B6rgers et al. [2], Edelman et al. [3], and Varian [17] have noted, this can be viewed as a continuous-time process in which bidders constantly readjust their bids to obtain the position that gives them the highest surplus. If the process stabilizes the result can then be modeled as a Nash equilibrium in pure strategies of the static one-shot game of complete information, since each bidder will be playing a best-response to the others' bids.²

²We do not claim that bidders will actually learn each others' private information, i.e. their true values and clickthrough rates, just that for a stable set of bids there is a corresponding equilibrium of the complete information

For the remainder of this section, we assume that there are as many positions as bidders.³ In a Nash equilibrium, each agent prefers his own position to the others given the bids, so the following inequalities must be satisfied:

$$\pi_i = \alpha_i \gamma_i (r_i - s_{i+1}) \quad (i \in N) \quad (3)$$

$$\pi_i \geq \alpha_i \gamma_j (r_i - s_{j+1}) \quad (i \in N, j > i) \quad (4)$$

$$\pi_i \geq \alpha_i \gamma_j (r_i - s_j) \quad (i \in N, j < i) \quad (5)$$

Here π_i can be interpreted as the agent in position i 's weighted utility. Agent i 's weighted utility for position j at price b per click is defined simply as $w_i u_i(j, b)$. Given the rules of the auction, for the allocation where agent i gets position i to actually arise, we also need the following conditions:

$$s_i \geq s_{i+1} \quad (1 \leq i \leq N-1) \quad (6)$$

$$s_i \geq 0 \quad (i \in N) \quad (7)$$

We should also require $\pi_i \geq 0$ for individual rationality, because a bidder always has the option of not participating in the auction, but this is in fact implied. For each bidder i we have

$$\pi_i \geq \alpha_i \gamma_n (r_i - s_{n+1}) = \alpha_i \gamma_n r_i \geq 0$$

and so $r_i \geq s_{i+1}$, i.e. the price paid by an agent in equilibrium is never greater than his value.

We ask the following question: given an allocation of positions to bidders together with bidder values, does there exist a vector of scores s such that inequalities (3)–(7) are satisfied? This gives us an idea of the allocations that can arise in equilibrium. Indeed, Börgers et al. [2] show that there can be a multitude of pure-strategy Nash equilibria in a position auction. The question can be answered using linear programming methods to test for the feasibility of the inequalities (3)–(7). Here we give some simple necessary conditions for the answer to be affirmative.

The following lemma shows that inequality (6) is not too restrictive.

Lemma 2 *If (π, s) satisfy inequalities (3)–(5), there exists (π', s') that satisfy inequalities (3)–(6) such that $s' \leq s$. Furthermore if (π, s) satisfy (7), then (π', s') satisfy (7).*

Proof. Assume we have a vector (π, s) which satisfies inequalities (3)–(5) but not (6). Then there is some i for which $s_i < s_{i+1}$. Construct a new vector (π, s') identical to the original except with $s'_{i+1} = s_i$. We now have $s'_i = s'_{i+1}$. An agent in position $j > i$ sees the price of position i decrease from s_{i+1}/w_j to $s'_{i+1}/w_j = s_i/w_j$, but this does not make i more preferred than j to this agent because we have

$$\pi_j \geq \gamma_{i-1} (r_j - s_i) \geq \gamma_i (r_j - s_i) = \gamma_i (r_j - s'_{i+1})$$

(i.e. because the agent in position j did not originally prefer position $i-1$ at price s_i/w_j , he will not prefer position i at price s_i/w_j). A similar argument applies for agents in positions $j < i$. Meanwhile, the agent in position i sees the price of its position go down, which only makes the position more preferred. Hence, if we set $\pi'_i = \alpha_i \gamma_i (r_i - s'_{i+1})$ and leave the remaining components of π unchanged, inequalities (3)–(5) remain valid for (π', s') . Repeating this process, we eventually

game.

³If there are more bidders than positions, we can add dummy positions with effects 0. If there are more positions than bidders, we can add dummy bidders each with weight 1 and value 0 per click. The analyses then proceed correctly.

obtain a vector that satisfies inequalities (3)–(6). It is clear by construction that if the initial vector is non-negative, the final vector is as well. $\square$

We can now derive necessary conditions for equilibrium allocations.

Proposition 1 *Taking π and s as the variables, inequalities (3)–(7) can be satisfied only if*

$$\left(1 - \frac{\gamma_i}{\gamma_{j+1}}\right) r_i \leq r_j$$

for $1 \leq j \leq N - 2$ and $i \geq j + 2$.

Proof. We safely ignore inequalities (6) in light of Lemma 2. For simplicity, we can also redefine π_i as π_i/α_i and the α_i drop out of the system of inequalities. By the Farkas lemma, the remaining inequalities can be satisfied if and only if there is no vector x such that

$$\begin{aligned} \sum_{i,j} (\gamma_j r_i) x_{ij} &> 0 \\ \sum_{i>j} \gamma_j x_{ij} + \sum_{i<j} \gamma_{j-1} x_{ij-1} &\leq 0 \quad (j \in N) \end{aligned} \tag{8}$$

$$\begin{aligned} \sum_j x_{ij} &\leq 0 \quad (i \in N) \\ x_{ij} &\geq 0 \quad (j \neq i) \\ x_{ii} &\text{ free} \end{aligned} \tag{9}$$

Now consider the following inequality, where $i > j$.

$$\frac{\gamma_j r_i}{\gamma_j} \leq \frac{\gamma_{j-1} r_{j-1}}{\gamma_{j-1}} + \frac{\gamma_i r_i}{\gamma_j} \tag{10}$$

If the inequality does not hold, then setting $x_{ij} = 1/\gamma_j$, $x_{j-1j-1} = -1/\gamma_{j-1}$, $x_{ii} = -1/\gamma_j$, and all other components of x to 0, we obtain a feasible solution with positive objective value. Hence inequality (10) must hold for inequalities (3)–(7) to be satisfied. By a slight reindexing, inequalities (10) for all $i > j$ yield the statement of the theorem. $\square$

Proposition 1 only gives necessary conditions for equilibrium allocations, not sufficient conditions, so it cannot be used to prove the existence of a Nash equilibrium. A pure-strategy Nash equilibrium exists in a position auction for any weighting scheme, but the proof of this fact is deferred to the next section. The import of Proposition 1 is the restriction it gives on possible equilibrium allocations; we apply the proposition in Section 3 to give a lower bound on equilibrium efficiency.

2.3 Symmetric Equilibrium

Varian [17] introduced a refinement of the Nash equilibrium concept for position auctions which he called “symmetric equilibrium.” Edelman et al. [3] independently introduced this refinement and called it “locally envy-free equilibrium.” With a slight modification we can make the Nash equilibrium inequalities above resemble those that arise in the assignment problem. In inequalities (5), we replace s_j by s_{j+1} . For clarity of notation we let $p_i = s_{i+1}$. A *symmetric* NE then satisfies

$$\pi_i = \alpha_i \gamma_i (r_i - p_i) \quad (i \in N) \tag{11}$$

$$\pi_i \geq \alpha_i \gamma_j (r_i - p_j) \quad (i \in N, j \neq i) \tag{12}$$

Varian [17] shows that symmetric equilibrium is indeed a refinement of Nash equilibrium. To confirm the existence of a symmetric (and hence Nash) equilibrium, we consider the following linear program

$$\begin{aligned} \min_{\pi, p} \quad & \sum_i \pi_i / \alpha_i + \sum_j \gamma_j p_j \\ \text{s.t.} \quad & \pi_i \geq \alpha_i \gamma_j (r_i - p_j) \quad \forall i, j \\ & \pi_i \geq 0 \quad \forall i \\ & p_j \geq 0 \quad \forall j \end{aligned}$$

The latter is the dual of the following program.

$$\begin{aligned} \max_x \quad & \sum_{i,j} (\gamma_j r_i) x_{ij} \\ \text{s.t.} \quad & \sum_j x_{ij} \leq 1 \quad \forall i \\ & \sum_i x_{ij} \leq 1 \quad \forall j \\ & x_{ij} \geq 0 \quad \forall i, j \end{aligned}$$

This is the linear programming formulation of a simple assignment problem which attempts to maximize the sum of the agents' scores, weighted by the position effects. It is well known that such programs have an integer optimal solution [16], which in this case explicitly describes the assignment of agents to positions. If x^* is an optimal assignment, and (π^*, x^*) is an optimal dual solution, then by complementary slackness we have

$$x_{ij}^* = 1 \Rightarrow \pi_i / \alpha_i = \gamma_j r_i - \gamma_j p_j$$

In words, assignment x^* and prices p^* constitute a symmetric equilibrium. Since the primal is obviously always feasible and bounded, the dual is as well, and there always exists a symmetric equilibrium.

It is not hard to characterize the assignment that maximizes $\sum_{i,j} (\gamma_j r_i) x_{ij}$. Because $\gamma_1 > \gamma_2 > \dots > \gamma_k$, it is optimal to rank agents in decreasing order of true scores, by standard results on rearrangements (see Hardy et al. [6] and also Lahaie [10]).

Because the agents have unit-demand valuations, the set of dual solutions to the assignment problem forms a convex lattice [16]. The maximal and minimal elements of this lattice are the solutions that maximize and minimize the component $\sum_j \gamma_j p_j$ of the objective, respectively. The maximal and minimal elements have simple closed-form expressions, namely the upper- and lower-recursive solutions for symmetric equilibria given by Varian [17] and Edelman et al. [3]. The form of the minimal element is particularly interesting:

$$\sum_{j=i+1}^n \alpha_i (\gamma_{j-1} - \gamma_j) \frac{w_j v_j}{w_i}. \quad (13)$$

(See Varian [17] or Lahaie and Pennock [11].) Note that payment (13) agrees exactly with (2). This is, of course, not an accident: Leonard [12] has shown that the minimal CE prices in the assignment problem coincide with Vickrey payments. The symmetric equilibrium concept states that agent bids should constitute CE prices. Hence minimal symmetric equilibrium bids should coincide with Vickrey payments, or, in the case of a position auction, the weighted equivalents of Vickrey

payments, which are given by Holmstrom's lemma. As a result, minimal symmetric equilibria and strategy-proof position auctions are revenue-equivalent, a point first made by Aggarwal et al. [1]. This observation will allow us to formulate the optimal position auction problem—where the objective is to maximize revenue, as opposed to efficiency—as a mathematical program in Section 4.

3 Efficiency

To maximize total value, we need to order the agents according to some permutation σ such that the inner product of the vectors $(\alpha_{\sigma(j)}v_{\sigma(j)})_{j \in K}$ and $(\gamma_j)_{j \in K}$ is maximized. As explained in the previous section, standard results on rearrangements then state that it is efficient to order the agents so that $\alpha_{\sigma(1)}\gamma_1v_{\sigma(1)} \geq \dots \geq \alpha_{\sigma(n)}\gamma_nv_{\sigma(n)}$.

When using the strategy-proof payment rule 2, agents reveal their true values and therefore are ranked by true score. Hence, if we take $w_i = \alpha_i$, the resulting equilibrium allocation is efficient. This fact applies to the symmetric equilibrium concept as well, because we saw in the previous section that in symmetric equilibrium, agents are ranked by true score.

According to Lemma 1, a ranking that results from a Nash equilibrium profile can only deviate from the allocation where agents are ranked by true score by having agents with relatively similar scores switch places. That is, if $r_i > r_j$, then agent j can be ranked higher than i only if the ratio r_j/r_i is sufficiently large. This suggests that the value of an equilibrium allocation cannot differ too much from the value obtained when agents are ranked by true score. The next result confirms this.

We denote the total value of an allocation σ of positions to agents by $f(\sigma) = \sum_{j=1}^k \gamma_j r_{\sigma(j)}$. Let

$$L = \min_{j=1, \dots, k-1} \min \left\{ \frac{\gamma_{j+1}}{\gamma_j}, 1 - \frac{\gamma_{j+2}}{\gamma_{j+1}} \right\}$$

(where by default $\gamma_{k+1} = 0$). Let η be the permutation such that $r_{\eta(1)} \geq \dots \geq r_{\eta(k)}$. Lahaie [10] gives the following bound.

Proposition 2 *For an allocation σ that results from a pure-strategy Nash equilibrium of a position auction, we have $f(\sigma) \geq Lf(\eta)$.*

Proof. We number the agents so that agent i has the i^{th} highest revenue, so $r_1 \geq r_2 \geq \dots \geq r_N$. Hence the standard allocation has value $f(\eta_r) = \sum_{i=1}^N \gamma_i r_i$. To prove the theorem, we will make repeated use of the fact that $\frac{\sum_t a_t}{\sum_t b_t} \geq \min_t \frac{a_t}{b_t}$ when the a_t and b_t are positive. Note that according to Proposition 1, if agent i lies at least two positions below position j , then $r_{\sigma(j)} \geq r_i \left(1 - \frac{\gamma_{j+2}}{\gamma_{j+1}}\right)$.

It may be the case that for some position i , we have $\sigma(i) > i$ and for positions $j > i + 1$ we have $\sigma(j) > i$. We then say that position i is *inverted*. Let S be the set of agents with indices at least $i + 1$; there are $n - i$ of these. If position i is inverted, it is occupied by some agent from S . Also all positions strictly lower than $i + 1$ must be occupied by the remaining agents from S , since $\sigma(j) > i$ for $j \geq i + 2$. The agent in position $i + 1$ must then have an index $\sigma(i + 1) \leq i$ (note this means position $i + 1$ cannot be inverted). Now there are two cases. In the first case we have

$\sigma(i) = i + 1$. Then

$$\begin{aligned} \frac{\gamma_i r_{\sigma(i)} + \gamma_{i+1} r_{\sigma(i+1)}}{\gamma_i r_i + \gamma_{i+1} r_{i+1}} &\geq \frac{\gamma_{i+1} r_i + \gamma_i r_{i+1}}{\gamma_i r_i + \gamma_{i+1} r_{i+1}} \\ &\geq \min \left\{ \frac{\gamma_{i+1}}{\gamma_i}, \frac{\gamma_i}{\gamma_{i+1}} \right\} \\ &= \frac{\gamma_{i+1}}{\gamma_i} \end{aligned}$$

In the second case we have $\sigma(i) > i + 1$. Then since all agents in S except the one in position i lie strictly below position $i + 1$, and the agent in position i is not agent $i + 1$, it must be that agent $i + 1$ is in a position strictly below position $i + 1$. This means that it is at least two positions below the agent that actually occupies position i , and by Proposition 1 we then have $r_{\sigma(i)} \geq r_{i+1} \left(1 - \frac{\gamma_{i+2}}{\gamma_{i+1}}\right)$. Thus,

$$\begin{aligned} \frac{\gamma_i r_{\sigma(i)} + \gamma_{i+1} r_{\sigma(i+1)}}{\gamma_i r_i + \gamma_{i+1} r_{i+1}} &\geq \frac{\gamma_{i+1} r_i + \gamma_i r_{\sigma(i)}}{\gamma_i r_i + \gamma_{i+1} r_{i+1}} \\ &\geq \min \left\{ \frac{\gamma_{i+1}}{\gamma_i}, 1 - \frac{\gamma_{i+2}}{\gamma_{i+1}} \right\} \end{aligned}$$

If position i is not inverted, then on one hand we may have $\sigma(i) \leq i$, in which case $r_{\sigma(i)}/r_i \geq 1$. On the other hand we may have $\sigma(i) > i$ but there is some agent with index $j \leq i$ that lies at least two positions below position i . Then by Proposition 1, $r_{\sigma(i)} \geq r_j \left(1 - \frac{\gamma_{i+2}}{\gamma_{i+1}}\right) \geq r_i \left(1 - \frac{\gamma_{i+2}}{\gamma_{i+1}}\right)$.

We write $i \in I$ if position i is inverted, and $i \notin I$ if neither i nor $i - 1$ are inverted. By our arguments above two consecutive positions cannot be inverted, so we can write

$$\begin{aligned} \frac{f(\sigma)}{f(\gamma_r)} &= \frac{\sum_{i \in I} (\gamma_i r_{\sigma(i)} + \gamma_{i+1} r_{\sigma(i+1)}) + \sum_{i \notin I} \gamma_i r_{\sigma(i)}}{\sum_{i \in I} (\gamma_i r_i + \gamma_{i+1} r_{i+1}) + \sum_{i \notin I} \gamma_i r_i} \\ &\geq \min \left\{ \min_{i \in I} \left\{ \frac{\gamma_i r_{\sigma(i)} + \gamma_{i+1} r_{\sigma(i+1)}}{\gamma_i r_i + \gamma_{i+1} r_{i+1}} \right\}, \min_{i \notin I} \left\{ \frac{\gamma_i r_{\sigma(i)}}{\gamma_i r_i} \right\} \right\} \\ &\geq L \end{aligned}$$

and this completes the proof. $\square$

For the common exponential decay model of $\gamma_i = \delta^{1-i}$ for $\delta > 1$, the factor becomes $L = \min\{\frac{1}{\delta}, 1 - \frac{1}{\delta}\}$. Feng et al. [4] report that an exponential decay model with $\delta = 1.428$ fits their Overture click-through rate data well. In this case, $L \approx 1/3.34$. Though being a factor of more than 3 away from the efficient value may seem unacceptable, we stress that this is a worst-case bound, and we would expect actual deviations to be much less than this in practice.

4 Revenue

Because multiple different allocations can arise in Nash equilibrium, it is difficult to give any bounds on equilibrium revenue. With the refinement of symmetric equilibrium, however, much more can be said. Here we provide a mathematical program whose solution gives the revenue-optimal ranking rule for a position auction given a distribution over bidder values are relevance, assuming

the minimal symmetric equilibrium is played. We focus on the minimal symmetric equilibrium, because optimizing revenue for this selection optimizes a lower bound on revenue in any symmetric equilibrium. The formulation also applies to dominant-strategy equilibrium revenue, because of the revenue equivalence between the two solution concepts.

We are interested in setting the weights w to achieve optimal expected revenue. The setup is as follows. The auctioneer chooses a function g so that the weighting scheme is $w_i \equiv g(\alpha_i)$. We do not consider weights that also depend on the agents' bids, because this would invalidate the equilibrium analysis of Section 2.3.⁴ A pool of n bidders is then obtained by i.i.d. draws of value-relevance pairs, where agent i 's type is drawn from the probability density $f_i(\alpha_i, v_i)$. We assume the densities are continuous and have full support on $[0, 1] \times [0, \infty)$, and we denote their marginals by f_i^α and f_i^v for relevance and value, respectively. The revenue to the auctioneer is then the revenue generated in minimal symmetric equilibrium under weighting scheme w . This assumes the auctioneer is patient enough not to care about revenue until bids have stabilized.

The problem of finding an optimal weighting scheme can be formulated as an optimization problem very similar to the one derived by Myerson [15] for the single-item auction case (with incomplete information). Let $Q_{ij}(\alpha, v; w) = 1$ if agent i obtains position j in equilibrium under weighting scheme w , where $\alpha = (\alpha_i)_{i \in N}$ and $v = (v_i)_{i \in N}$ are the realized relevance and value vectors, and let it be 0 otherwise.

We parallel Krishna's approach to the single-item optimal auction formulation (see chapter 5 of Krishna [9]). Let $q_i(\alpha_i, v_i; w)$ be the expected click-through rate that bidder i receives in minimal symmetric equilibrium when his relevance is α_i and his value v_i , and the weighting scheme is w :

$$q_i(\alpha_i, v_i; w) = \int_{[0,1]^{n-1}} \int_{[0,\infty]^{n-1}} \sum_{j=1}^n \alpha_i \gamma_j Q_{ij}(\alpha_i, \alpha_{-i}, v_i, v_{-i}; w) f_{-i}(\alpha_{-i}, v_{-i}) dv_{-i} d\alpha_{-i}$$

Recall from the discussion in Section 2.3 that the total payment of an agent in minimal symmetric equilibrium is the payment derived by the application of Holmstrom's lemma in Section 2.1, which is

$$\sum_{j=1}^n \alpha_i \gamma_j v_i Q_{ij}(\alpha_i, \alpha_{-i}, v_i, v_{-i}; w) - \int_0^{v_i} \sum_{j=1}^n \alpha_i \gamma_j Q_{ij}(\alpha_i, \alpha_{-i}, \tau_i, v_{-i}; w) d\tau_i.$$

The expected payment of agent i in minimal symmetric equilibrium is therefore

$$q_i(\alpha_i, v_i; w) v_i - \int_0^{v_i} q_i(\alpha_i, \tau_i; w) d\tau_i.$$

The analysis now proceeds almost identically to the single-item case. The *ex ante* expected payment of bidder i is

⁴The analysis does not generalize to weights that depend on bids. It is unclear whether an equilibrium would exist at all with such weights.

$$\begin{aligned}
& \int_0^1 \int_0^\infty \left(q_i(\alpha_i, v_i; w) v_i - \int_0^{v_i} q_i(\alpha_i, \tau_i; w) d\tau_i \right) f_i(\alpha_i, v_i) dv_i d\alpha_i \\
&= \int_0^1 \int_0^\infty q_i(\alpha_i, v_i; w) v_i f_i(\alpha_i, v_i) dv_i d\alpha_i - \int_0^1 \int_0^\infty \int_0^{v_i} q_i(\alpha_i, \tau_i; w) f_i(\alpha_i, v_i) d\tau_i dv_i d\alpha_i \\
&= \int_0^1 \int_0^\infty q_i(\alpha_i, v_i; w) v_i f_i(\alpha_i, v_i) dv_i d\alpha_i - \int_0^1 \int_0^\infty \int_{\tau_i}^\infty q_i(\alpha_i, \tau_i; w) f_i(\alpha_i, v_i) dv_i d\tau_i d\alpha_i \\
&= \int_0^1 \int_0^\infty q_i(\alpha_i, v_i; w) v_i f_i(v_i|\alpha_i) f_i^\alpha(\alpha_i) dv_i d\alpha_i \\
&\quad - \int_0^1 \int_0^\infty (1 - F_i(\tau_i|\alpha_i)) q_i(\alpha_i, \tau_i; w) f_i^\alpha(\alpha_i) d\tau_i d\alpha_i \\
&= \int_0^1 \int_0^\infty \left(v_i - \frac{1 - F_i(v_i|\alpha_i)}{f_i(v_i|\alpha_i)} \right) q_i(\alpha_i, v_i; w) f_i(v_i|\alpha_i) dv_i d\alpha_i \\
&= \int_{[0,1]^n} \int_{[0,\infty]^n} \sum_{j=1}^n \alpha_i \gamma_j \left(v_i - \frac{1 - F_i(v_i|\alpha_i)}{f_i(v_i|\alpha_i)} \right) Q_{ij}(\alpha, v; w) f(\alpha, v) dv d\alpha
\end{aligned}$$

where the second equality follows by interchanging the order of integration in the second term, the third by the fact that $f_i(\alpha_i, v_i) = f_i(v_i|\alpha_i) f_i^\alpha(\alpha_i)$, and the fifth by the definition of $q_i(\alpha_i, v_i; w)$. We define agent i 's "virtual valuation" by

$$\psi_i(\alpha_i, v_i) = v_i - \frac{1 - F_i(v_i|\alpha_i)}{f_i(v_i|\alpha_i)}.$$

Summing over each agent's *ex ante* expected payment, the expected revenue is then

$$\int_{[0,1]^n} \int_{[0,\infty]^n} \sum_{i=1}^n \sum_{j=1}^n \alpha_i \gamma_j \psi_i(\alpha_i, v_i) Q_{ij}(\alpha, v; w) f(\alpha, v) dv d\alpha$$

According to this analysis, we should rank bidders by "virtual score" $\alpha_i \psi(\alpha_i, v_i)$ to optimize revenue (and exclude any bidders with negative virtual score). However, unlike in the incomplete information setting, here we are constrained to ranking rules that correspond to a certain weighting scheme $w_i \equiv g(\alpha_i)$. Lahaie and Pennock [11] show that there is no function g such that $\alpha_i \psi(\alpha_i, v) = g(\alpha_i) v$, for any distribution f ; i.e. the virtual score is never a linear function of value, for fixed effect α_i .

Of course, to rank bidders by virtual score, we only need $g(\alpha_i) v_i = h(\alpha_i \psi(\alpha_i, v_i))$ for some monotonically increasing transformation h . (A necessary condition for this is that $\psi(\alpha_i, v_i)$ be increasing in v_i for all α_i .) Still, it is unclear how to do this in general. Lahaie and Pennock [11] study the restricted family of weights $w_i = \alpha_i^q$ for $q \in (-\infty, +\infty)$ and note that smaller q imply greater revenue when value and relevance are correlated. This kind of approach, where restricted families of weights are considered rather than arbitrary functions g , appears to be a fruitful avenue for progress towards the revenue-optimal mechanism.

Our analysis does allow for an analytical solution in one special case. Suppose that value is independent of relevance, and that the distribution over value is uniform on $[0, 1]$. Then the virtual

score of a bidder is $2\alpha_i v_i - \alpha_i$. We can recreate this by giving each bidder a weight of $2\alpha_i$, and introducing a discount of α_i . (The discount can be construed as a kind of reserve score, because a bidder whose score $\alpha_i v_i$ does not exceed the discount should not be shown.) The symmetric equilibrium analysis generalizes to the case where additive discounts are included in the score, so this ranking scheme is indeed revenue-optimal when value is uniformly distributed.

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