CS6998-3: Solutions to Problem Set # 2

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Problem 1 (10 points)

(a) **5 points**

The proof is virtually identical to the proof that the VCG mechanism is truthful. Let v_{-i} be some *arbitrary* reports by agents $N-i$, let v_i be agent i's true valuation, and let $\tilde{v}_i$ be some other valuation. Let R be an efficient allocation with respect to (v_i, v_{-i}), and let R' be an efficient allocation with respect to $(\tilde{v}_i, v_{-i})$. If agent i reports v_i its utility is

$$\sum_{j \in N} v_j(R_j) - h_i(v_{-i}), \tag{1}$$

whereas if it reports $\tilde{v}_i$ its utility is

$$\sum_{j \in N} v_j(R'_j) - h_i(v_{-i}). \tag{2}$$

Subtracting (2) from (1) we get

$$\sum_{j \in N} v_j(R_j) - \sum_{j \in N} v_j(R'_j).$$

This is non-negative because R is efficient with respect to the profile (v_i, v_{-i}). Thus reporting v_i maximizes i's utility, since $\tilde{v}_i$ was arbitrary.

(b) **5 points**

By definition, a Groves mechanism is individually rational if the utility to each agent i from truthfully reporting its value is non-negative. Let R be the efficient allocation selected if i reports truthfully.

The utility to agent i is

$$\sum_{j \in N} v_j(R_j) - h_i(v_{-i}),$$

so we must have

$$h_i(v_{-i}) \leq \sum_{j \in N} v_j(R_j). \tag{3}$$

This holds for any possible valuation of agent i, in particular the valuation where $v_i(S) = 0$ for all $S \subseteq M$. In this case we can assume that $R_i = \emptyset$, and thus R is an efficient allocation among agents $N - i$. Condition (3) in this special case is

$$h_i(v_{-i}) \leq \max_{R' \in \Gamma} \sum_{j \in N-i} v_j(R'_j). \tag{4}$$

The Groves mechanism that maximizes the term $h_i(v_{-i})$ is the one that maximizes agent i's payment. In view of (4), the VCG mechanism maximizes the payment because it achieves the upper bound.

Problem 2 (10 points)

(a) **7 points**

We first show that if R is an efficient allocation, then

$$v_i(R_i) = \max_{j \in N} v_j(R_j). \tag{5}$$

Assume this does not hold, so that for some $i \in N$, there is a $k \neq i$ such that $v_k(R_i)$ is the maximum value for R_i over all agents. Note that this value must be positive.

As $v_k(R_i) > 0$, we have $R_i \supseteq S_k$. However, $R_i \cap R_k = \emptyset$ by the feasibility of R. Thus $R_k \not\supseteq S_k$ and $v_k(R_k) = 0$. Suppose that instead of giving R_i to i and R_k to k, we give R_i to k and $\emptyset$ to i. Then this changes the total value by $v_k(R_i) - v_i(R_i) > 0$. This is a contradiction because R is efficient.

Hence $v_i(R_i) - p(R_i) = 0$ for each $i \in N$, and $v_i(S) - p(S) \leq 0$ by the definition of p; the bundle R_i maximizes i's utility, for all $i \in N$. It remains for us to show that R maximizes revenue at prices p.

Let R' be a revenue-maximizing allocation such that the number of agents that receive $\emptyset$ is maximized. Note that if we permute the bundles in R', the revenue remains unchanged, because prices are anonymous. For each R'_i, let $\sigma(i) \in N$ be an agent such that $p(R'_i) = v_{\sigma(i)}(R'_i)$. We claim that we must have $\sigma(i) \neq \sigma(j)$ when $R'_i \neq \emptyset$ and $R'_j \neq \emptyset$. Assume for the sake of contradiction that $p(R'_i) + p(R'_j) = v_k(R'_i) + v_k(R'_j)$. Since $R'_i \cap R'_j = \emptyset$ and v_k is single-minded, the value of one of those bundles to agent k must be 0, say $v_k(R'_j) = 0$. Thus,

$$\begin{aligned} v_k(R'_i) + v_k(R'_j) &\leq v_k(R'_i \cup R'_j) + v_k(\emptyset) \\ &\leq p(R'_i \cup R'_j) + p(\emptyset). \end{aligned}$$

We see that if we replace R'_i with $R'_i \cup R'_j$, and R'_j with $\emptyset$, we get an allocation R'' with weakly greater revenue than R'. But since the latter is revenue-maximizing, so is R''. This is a contradiction, because R'' contains one more $\emptyset$ than R. Thus we can permute the bundles in R' such that $p(R'_i) = v_i(R'_i)$ for $R'_i \neq \emptyset$. For $R'_i = \emptyset$, we have $p(\emptyset) = v_j(\emptyset)$ for all $j \in N$. After the permutation, the revenue from R' is

$$\begin{aligned} \sum_{i \in N} p(R'_i) &= \sum_{i \in N} v_i(R'_i) \\ &\leq \sum_{i \in N} v_i(R_i) \\ &\leq \sum_{i \in N} p(R_i) \end{aligned}$$

where the second step follows because R is efficient, and the third from (5). As R' is revenue-maximizing, so is R, and this completes the proof.

(b) **3 points**

Let $\Gamma(S)$ be the set of all feasible allocations such that $R_i = S$ for some $i \in N$. We have a variable $x_i(S)$ for each $i \in N$ and $S \subseteq M$ to denote whether i obtains bundle S. We have a variable $z(R)$ for each feasible allocation R to denote whether R is selected.

$$\begin{aligned} \max_{x \geq 0, z \geq 0} \quad & \sum_{i \in N} \sum_{S \subseteq M} v_i(S) x_i(S) \\ \text{subject to} \quad & \sum_{i \in N} x_i(S) = \sum_{R \in \Gamma(S)} z(R) \quad (S \subseteq M) \\ & \sum_{S \subseteq M} x_i(S) = 1 \qquad (i \in N) \\ & \sum_{R \in \Gamma} z(R) = 1 \end{aligned}$$

Problem 3 (10 points)

(a) **3 points**

First note that any allocation that does not allocate exactly one item to each agent must be inefficient. If some item is unallocated, some agent obtains nothing, and giving this agent the unallocated item strictly increases the total value. If some agent i obtains more than one item, at least one other agent j must receive nothing. If we give agent i's least preferred item to agent j, this strictly increases the total value. Thus we can restrict our attention to one-to-one allocations of the items to the agents.

Consider any one-to-one allocation besides that where agent i gets item i for each $i \in N$. We will show that it is not efficient. Let j be the lowest-indexed agent that does not receive item j. Let k be the item that j receives; note that by our assumption on j, we must have $k > j$. Let $\ell > j$ be the agent that receives item j. If we swap the items, giving item j to agent j and item k to agent ℓ, the difference between the total value of this new allocation and that of the original allocation is

$$(a_j b_j + a_\ell b_k) - (a_j b_k + a_\ell b_j) = (a_j - a_\ell)(b_j - b_k) > 0.$$

This is positive because $j < \ell$ and $j < k$. Thus the allocation was not efficient.

(b) **3 points**

The first set of of inequalities implies the second, because the second is a subset of the first. We show that the second set implies the first.

Note that $v_{ii} - p_i \geq v_{ii-1} - p_{i-1}$ implies

$$p_{i-1} - p_i \geq v_{ii-1} - v_{ii} = a_i(b_{i-1} - b_i) > 0.$$

Thus $p_1 > p_2 > \ldots > p_n$, and $p_n \geq 0$ implies $p_i \geq 0$ for all $i \in M$. We now prove by induction that $v_{ii} - p_i \geq v_{ii+1} - p_{i+1}$ implies $v_{ii} - p_i \geq v_{ij} - p_j$ for all $j > i$. The base case $j = i + 1$ holds by assumption. Assume the claim holds for $j - 1 > i$:

$$v_{ii} - v_{ij-1} \geq p_i - p_{j-1}. \tag{6}$$

Now since $v_{j-1j-1} - v_{j-1j} \geq p_{j-1} - p_j$ and $v_{j-1j-1} - v_{j-1j} = a_{j-1}(b_{j-1} - b_j) < a_i(b_{j-1} - b_j) = v_{ij-1} - v_{ij}$, we have

$$v_{ij-1} - v_{ij} \geq p_{j-1} - p_j. \tag{7}$$

Adding (6) and (7) proves the claim. The proof that $v_{ii} - p_i \geq v_{ii-1} - p_{i-1}$ implies $v_{ii} - p_i \geq v_{ij} - p_j$ for all $j < i$ is entirely analogous.

(c) **1 point**

Let p and p' be first-order competitive equilibrium prices. Since $p \geq \mathbf{0}$ and $p' \geq \mathbf{0}$, we clearly have $p \wedge p' \geq \mathbf{0}$. To prove that $p \wedge p'$ is a CE price vector we need to show that for all i, j

$$\min\{p_i, p_i'\} - \min\{p_j, p_j'\} \leq v_{ii} - v_{ij}.$$

Assume without loss of generality that $\min\{p_j, p_j'\} = p_j$. Then we have the following derivation.

$$\begin{aligned} p_i - p_j &\leq v_{ii} - v_{ij} \\ p_i - \min\{p_j, p_j'\} &\leq v_{ii} - v_{ij} \\ \min\{p_i, p_i'\} - \min\{p_j, p_j'\} &\leq v_{ii} - v_{ij} \end{aligned}$$

(d) **2 points**

It is straightforward to check that $\bar{p}$ satisfy the inequalities of part (b), so they are competitive equilibrium prices. We prove that they are minimal by induction. Let p be first-order CE prices. We have $p_n \geq 0$ by definition, and note that $\bar{p}_n = 0$. Thus $p_n \geq \bar{p}_n$, establishing the base case.

Assume $p_i \geq \bar{p}_i$ where $i \leq n$. For $i = 2, \ldots, n$, we have $v_{ii} - p_i \geq v_{ii-1} - p_{i-1}$ which implies

$$\begin{aligned} p_{i-1} &\geq v_{ii-1} - v_{ii} + p_i \\ &\geq v_{ii-1} - v_{ii} + \bar{p}_i \\ &= a_i(b_{i-1} - b_i) + \sum_{j>i} a_j(b_{j-1} - b_j) \\ &= \sum_{j>i-1} a_j(b_{j-1} - b_j) \\ &= \bar{p}_{i-1}. \end{aligned}$$

The second inequality follows from the induction hypothesis, and the remaining from the definition of v_{ii-1} and $\bar{p}_i$. This completes the proof.

(e) **1 point**

Fix agent i. With all agents present, the efficient allocation gives item 1 to agent 1, item 2 to agent 2, etc. by part (a). The total value to all the agents except i under this allocation is

$$\sum_{j \neq i} v_{jj} = \sum_{j \neq i} a_j b_j. \tag{8}$$

If agent i is removed, the efficient allocation gives item j to agent j for $j < i$, and item $j - 1$ to agent j for $j > i$ (item n remains unallocated). This follows from the same reasoning as in part (a). The total value to all the agents except j in this case is

$$\sum_{j<i} v_{jj} + \sum_{j>i} v_{jj-1} = \sum_{j<i} a_j b_j + \sum_{j>i} a_j b_{j-1}. \tag{9}$$

By definition the VCG payment of agent i is (9) minus (8):

$$\begin{aligned} \hat{q}_i &= \sum_{j<i} a_j b_j + \sum_{j>i} a_j b_{j-1} - \sum_{j \neq i} a_j b_j \\ &= \sum_{j>i} a_j(b_{j-1} - b_j). \end{aligned}$$

Comparing with part (d), we find that $\hat{q}_i = \bar{p}_i$. That is, the VCG payment of agent i is the price of the item agent i receives at the lowest possible linear CE prices.