

CS6998-3: Solutions to Problem Set # 1

Etienne Vouga

Problem 1 (10 points)

- (a) We know from the theorem stated (but not proven) in class that it suffices to restrict our attention to two-link, two-node graphs, with edge latency functions $l_1(x) = ax^i$ and $l_2(x) = 1$. In other words, letting $P_d(N)$ denote the price of anarchy of a graph N with edge latencies in $\mathcal{M}_d$, and $[f(x); g(x)]$ a two-link, two-node graph with latency functions $f(x)$ and $g(x)$, we have

$$P_d(N) \leq \max_{a, 0 \leq i \leq d} P_d([ax^i; 1]).$$

We now compute the right-hand side in five steps.

Lemma 0.1. *If both latency functions are constant, Nash and optimal flows have equal cost:*

$$\max_a P_d([ax^0; 1]) = 1.$$

Proof. If both latency functions are constant, obviously the flow routing all supply through the edge with least latency is both optimal and Nash. $\square$

Lemma 0.2. *If $i \neq 0$ and $a \leq (1+i)^{-1}$, Nash and optimal flows have equal cost:*

$$\max_{a \leq (1+i)^{-1}, 1 \leq i \leq d} P_d([ax^i; 1]) = 1.$$

Proof. We check that the flow S routing all supply through the first edge is both optimal and Nash. Since S sends no supply through the second edge, S is optimal if the marginal cost of the first edge is at most that of the second:

$$\begin{aligned} a &\leq (i+1)^{-1} \\ a(i+1) &\leq 1 \\ [a(i+1)x^i](1) &\leq 1 \\ \frac{d}{dx}(xax^i)(1) &\leq \frac{d}{dx}(x)(0) \\ c'_1(1) &\leq c'_2(0), \end{aligned}$$

as required. Thus S is optimal.

To check S is Nash, we check that the latency of the first edge is at most the latency of the second:

$$\begin{aligned} a &\leq 1 \\ [ax^i](1) &\leq [1](0) \\ l_1(1) &\leq l_2(0), \end{aligned}$$

so S is also Nash. $\square$

Lemma 0.3. *If $i \neq 0$ and $a > 1$,*

$$\max_{a>1, 1 \leq i \leq d} P_d([ax^i; 1]) = \frac{1}{1 - d(d+1)^{-(d+1)/d}}.$$

Proof. Since every step of the inequality manipulation in Lemma 0.2 is reversible, and $a > 1$ (and thus $a > (1+i)^{-1}$), we know the flow routing all supply through the first edge is neither Nash nor optimal. We also check briefly that the flow routing all supply through the second edge is neither optimal nor Nash:

The marginal costs for such a flow are

$$c'_1(0) = [(i+1)ax^i](0) = 0$$

since $i \neq 0$, and

$$c'_2(1) = 1,$$

so $c'_2(1) \not\leq c'_1(0)$ and such flow is not optimal. The latencies are

$$l_1(0) = [ax^i](0) = 0$$

$$l_2(1) = 1$$

so similarly $l_2(1) \not\leq l_1(0)$ and such flow is not Nash.

We conclude that both the Nash and optimal flow sends a little bit of supply through the first edge, and a little bit through the second. Thus, writing S^* for the Nash flow, $C(S^*)$ for the cost of the Nash flow, and y as the amount of supply sent through the first edge, we have equality of latencies:

$$\begin{aligned} l_1(y) &= l_2(1-y) \\ ay^i &= 1. \end{aligned} \tag{1}$$

The formula for the $C(S^*)$ is

$$C(S^*) = yay^i + (1-y). \tag{2}$$

Plugging (1) into (2) gives

$$C(S^*) = y + (1-y) = 1.$$

What about the cost of the optimal solution? Let S be the optimal solution, which sends z supply through the first edge. We have equality of marginal costs:

$$\begin{aligned} c'_1(z) &= c'_2(z) \\ (i+1)az^i &= 1, \end{aligned} \tag{3}$$

and a formula for $C(S)$,

$$C(S) = zaz^i + (1-z). \tag{4}$$

Solving for z in (3) yields

$$z = [a(i+1)]^{-1/i}.$$

Substituting (3) into (4) gives

$$\begin{aligned} C(S) &= \frac{z}{i+1} + (1-z) \\ &= 1 - \frac{i}{i+1}z \\ &= 1 - \frac{i}{i+1}[a(i+1)]^{-1/i}. \end{aligned}$$

Thus we have the price of anarchy,

$$\frac{C(S^*)}{C(S)} = \frac{1}{1 - \frac{i}{i+1} [a(i+1)]^{-1/i}}.$$

What values of i and a maximize the price of anarchy? a is easy: decreasing a increases $[a(i+1)]^{-1/i}$, which decreases $C(S)$, which increases the price of anarchy, so

$$\max_{a>1} P_d([ax^i; 1]) = \frac{1}{1 - \frac{i}{i+1} [(i+1)]^{-1/i}} = \frac{1}{1 - i[(i+1)]^{-(1+i)/i}}.$$

To find the right value of d , we notice that maximizing P_d is equivalent to maximizing $f(i) = i[(i+1)]^{-(1+i)/i}$. We take the derivative $f'(i)$:

$$\begin{aligned} \log f(i) &= \log(i) - \frac{1+i}{i} \log(i+1) \\ \frac{1}{f(i)} f'(i) &= \frac{1}{i} + \frac{1}{i^2} \log(i+1) - \frac{1}{i} \\ f'(i) &= f(i) \left(\frac{1}{i^2} \log(i+1) \right) \\ &> 0 \end{aligned}$$

since $i > 0$, so we maximize f by maximizing i :

$$\max_{a>1, 1 \leq i \leq d} P_d([ax^i; 1]) = \frac{1}{1 - d(d+1)^{-(d+1)/d}}.$$

□

We have one case left:

Lemma 0.4. *If $i \neq 0$ and $(1+i)^{-1} < a \leq 1$,*

$$\max_{(1+i)^{-1} < a \leq 1, 1 \leq i \leq d} P_d([ax^i; 1]) = \frac{1}{1 - d(d+1)^{-(d+1)/d}}.$$

Proof. We know from the previous lemmas that for $a > (1+i)^{-1}$, the optimal flow has cost

$$C(S) = 1 - \frac{i}{i+1} [a(i+1)]^{-1/i},$$

and that for $a \leq 1$ the Nash flow C^* sends all supply through the first edge. The cost of the Nash flow is then

$$C(S^*) = [xax^i](1) + [1](0) = a.$$

The price of anarchy is thus

$$\frac{C(S^*)}{C(S)} = \frac{a}{1 - \frac{i}{i+1} [a(i+1)]^{-1/i}} = f(a).$$

¹I'm being liberal with notation here... technically max should be sup.

Unlike in the previous lemma, it is no longer obvious how changing a changes f , so we resort to taking a derivative using the quotient rule:

$$\begin{aligned}
f'(a) &= \frac{1 - \frac{i}{i+1}[a(i+1)]^{-1/i} - a[a(i+1)]^{-1/i-1}}{\left(1 - \frac{i}{i+1}[a(i+1)]^{-1/i}\right)^2} \\
&= \frac{1 - \frac{i}{i+1}[a(i+1)]^{-1/i} - \frac{1}{i+1}[a(i+1)]^{-1/i}}{\left(1 - \frac{i}{i+1}[a(i+1)]^{-1/i}\right)^2} \\
&= \frac{1 - [a(i+1)]^{-1/i}}{\left(1 - \frac{i}{i+1}[a(i+1)]^{-1/i}\right)^2} \\
&> 0
\end{aligned}$$

since $a(i+1) > 1$ and $i \geq 1$. So we maximize the price of anarchy by maximizing a , which in this case means setting $a = 1$:

$$\max_{(1+i)^{-1} < a \leq 1} P_d([ax^i; 1]) = \frac{1}{1 - i(i+1)^{-(i+1)/i}},$$

the same expression we had in the previous lemma, so as before, to get an upper bound we set $i = d$, and

$$\max_{(1+i)^{-1} < a \leq 1, 1 \leq i \leq d} P_d([ax^i; 1]) = \frac{1}{1 - d(d+1)^{-(d+1)/d}}.$$

□

Finally, we put all of the above lemmas together to get

$$\max_{0 \leq i \leq d} P_d([ax^i; 1]) = \max\left(1, \frac{1}{1 - d(d+1)^{-(d+1)/d}}\right) = \frac{1}{1 - d(d+1)^{-(d+1)/d}}.$$

As a sanity check, we notice that this bound does reduce to $\frac{4}{3}$ for the special case $d = 1$.

- (b) Again, we can restrict our attention to two-link, two-node networks $[f_d(x); 1]$, where $f_d(x) = \sum_{i=0}^d a_i x^i$ is a polynomial of degree at most d . But this network is equivalent to a two-node, $d+2$ -link network consisting of two paths: the first links the source and the sink with latency 1. The second is a chain of $d+1$ links, whose i th link (starting at $i = 0$) has latency $a_i x^i$. This new network is one whose latencies are all monomial in $\mathcal{M}_d$, so by Part A we conclude

$$\max_{0 \leq i \leq d} P_d([f_d(x); 1]) \leq \frac{1}{1 - d(d+1)^{-(d+1)/d}}.$$

Since $\mathcal{M}_d \subset \mathcal{P}_d$, this bound is still tight, so

$$\max_{0 \leq i \leq d} P_d([f_d(x); 1]) = \frac{1}{1 - d(d+1)^{-(d+1)/d}}.$$

Problem 2 (10 points)

- (a) Let S be a viable solution, with n paths $p_{i,j}$ routing $x_{i,j}$ units of supply from s_i to t_i , with $\sum_j x_{i,j} = 1$. Suppose S is a Nash solution, with $p_{i,j}$ any path in S with $x_{i,j} > 0$, and consider a second path p'_i from s_i to t_i . Since the supply travelling through $p_{i,j}$ is acting greedily, the latency of the $p_{i,j}$ must be at most that of p'_i , or some supply would switch to flowing through there instead. Thus

$$\sum_{e \in p_{i,j}} l_e(x_e) \leq \sum_{e \in p'_i} l_e(x_e).$$

where l_e is the latency function of edge e , and x_e the total amount of supply (from all sources) flowing through e . Since $l_e(x) = a_e x + b_e$ is linear,

$$\sum_{e \in p_{i,j}} a_e x_e + b_e \leq \sum_{e \in p'_i} a_e x_e + b_e. \quad (5)$$

Now suppose S is optimal, and again let $p_{i,j}$ be any path with $x_{i,j} > 0$. Optimality means transferring any amount δ of supply from $p_{i,j}$ to another path p'_i from s_i to t_i cannot improve the total cost of S , so the change in cost of such a switch must be nonnegative.

Consider the edges e on $p_{i,j}$. We can partition such edges into two sets: those edges U also on the path p'_i , and those V_1 that are not. Let V_2 be the set of edges of p'_i not on $p_{i,j}$, that is, those not in U . Then, writing $c_e(x) = x l_e(x)$, the change in cost of switching δ supply from $p_{i,j}$ to p'_i is

$$\sum_{e \in V_1} c_e(x_e) + \sum_{e \in V_2} c_e(x_e) - \sum_{e \in V_1} c_e(x_e - \delta) - \sum_{e \in V_2} c_e(x_e + \delta).$$

Thus

$$\sum_{e \in V_1} c_e(x_e) + \sum_{e \in V_2} c_e(x_e) - \sum_{e \in V_1} c_e(x_e - \delta) - \sum_{e \in V_2} c_e(x_e + \delta) \leq 0.$$

We now proceed exactly as in the lecture notes. Manipulating this equation, and applying the definition of the derivative, yields

$$\begin{aligned} \sum_{e \in V_1} c_e(x_e) - \sum_{e \in V_1} c_e(x_e - \delta) &\leq \sum_{e \in V_2} c_e(x_e + \delta) - \sum_{e \in V_2} c_e(x_e) \\ \sum_{e \in V_1} [c_e(x_e) - c_e(x_e - \delta)] &\leq \sum_{e \in V_2} [c_e(x_e + \delta) - c_e(x_e)] \\ \sum_{e \in V_1} [c_e(x_e) - c_e(x_e - \delta)] + \sum_{e \in U} [c_e(x_e + \delta) - c_e(x_e)] &\leq \sum_{e \in V_2} [c_e(x_e + \delta) - c_e(x_e)] + \sum_{e \in U} [c_e(x_e + \delta) - c_e(x_e)] \\ \sum_{e \in V_1} \frac{c_e(x_e) - c_e(x_e - \delta)}{\delta} + \sum_{e \in U} \frac{c_e(x_e + \delta) - c_e(x_e)}{\delta} &\leq \sum_{e \in V_2} \frac{c_e(x_e + \delta) - c_e(x_e)}{\delta} + \sum_{e \in U} \frac{c_e(x_e + \delta) - c_e(x_e)}{\delta} \\ \sum_{e \in V_1} c'_e(x_e) + \sum_{e \in U} c'_e(x_e) &\leq \sum_{e \in V_2} c'_e(x_e) + \sum_{e \in U} c'_e(x_e) \\ \sum_{e \in p_{i,j}} c'_e(x_e) &\leq \sum_{e \in p'_i} c'_e(x_e). \end{aligned}$$

Since l_e is linear, $c_e(x_e) = a_e x_e^2 + b_e x_e$, and $c'_e(x_e) = 2a_e x_e + b_e$, so

$$\sum_{e \in p_{i,j}} 2a_e x_e + b_e \leq \sum_{e \in p'_i} 2a_e x_e + b_e. \quad (6)$$

(b) Let S be a Nash solution. Then by (5),

$$\sum_{e \in p_{i,j}} a_e x_e + b_e \leq \sum_{e \in p'_i} a_e x_e + b_e.$$

Now consider the flow S' found by halving the amount of supply flowing through each path in S . This flow routes $\frac{1}{2}$ of a unit from each s_i to each t_i , and if x_e and x'_e are the total supply passing through an edge for the flow S and S' respectively, $2x'_e = x_e$. Substituting into the above equation gives

$$\sum_{e \in p_{i,j}} 2a_e x'_e + b_e \leq \sum_{e \in p'_i} 2a_e x'_e + b_e,$$

which is exactly (6). Thus S' is optimal for routing $\frac{1}{2}$ of a unit of supply from the sources to the sinks.

Problem 3 (10 points)

Let S^* be a Nash solution, S an optimal solution, $C(T)$ the cost of a viable solution T , and assume, for contradiction, that

$$\frac{C(S^*)}{C(S)} > k.$$

Denote by $c_i(S^*)$ the cost charged to player i for S^* . Let player j be the player charged the most; that is,

$$j = \arg \max_i c_i(S^*).$$

Then

$$\begin{aligned} C(S^*) &= \sum_i c_i(S^*) \\ &\leq k c_j(S^*) \\ c_j(S^*) &\geq \frac{C(S^*)}{k}. \end{aligned}$$

Now consider player j switching to whatever path he uses in S , yielding a new viable solution S' . $c_j(S')$ is no more expensive than unilaterally buying all edges in S : $c_j(S') \leq C(S)$. Then

$$\begin{aligned} c_j(S') &\leq C(S) \\ &< \frac{C(S^*)}{k} \\ &\leq c_j(S^*), \end{aligned}$$

so $c_j(S') < c_j(S^*)$, a contradiction since S^* is a Nash solution.

Problem 4 (10 points)

(a) We begin with a technical lemma:

Lemma 0.5. $H_k \leq \log k + 1$ for $k \geq 2$.

Proof. Since $\frac{d}{dx} \left(\frac{1}{x} \right) < 0$, the sum

$$\sum_{i=2}^k \frac{1}{i} = H_k - 1$$

is a right Riemann sum of $f(x) = \frac{1}{x}$ from 1 to k , which underestimates $\int_1^k \frac{1}{x} = \log k$. Thus

$$\begin{aligned} H_k - 1 &\leq \log k \\ H_k &\leq \log k + 1. \end{aligned}$$

□

Now let S be any viable starting solution to the network design game. The greatest possible value of $\Phi(S)$ is $C|E|H_k \leq C|E|(\log k + 1)$. Then if S^j is the solution after j steps of making large changes to the potential,

$$\Phi(S^j) \leq \left(1 - \frac{\epsilon}{k}\right)^j C|E|(\log k + 1).$$

The least non-zero value of Φ is 1, so if we ever have $\Phi \leq 1$, we know we will reach $\Phi = 0$, which must be a Nash equilibrium, in the next step. Thus, to bound the worst-case number of steps s needed to reach an equilibrium, we solve

$$1 = \left(1 - \frac{\epsilon}{k}\right)^j C|E|(\log k + 1), \quad (7)$$

and know $s \leq 1 + j$.

Taking the logarithm of both sides of (7) gives

$$\begin{aligned} 0 &= j \log \left(1 - \frac{\epsilon}{k}\right) + \log C + \log |E| + \log(\log k + 1) \\ j &= \frac{\log C + \log |E| + \log(\log k + 1)}{\log \left(\frac{k}{k-\epsilon}\right)} \\ s &\leq 1 + \frac{\log C + \log |E| + \log(\log k + 1)}{\log k - \log(k - \epsilon)}. \end{aligned}$$

To show s is polynomial in several variables, it is enough to show that it is polynomial in each individual variable with the other variables treated as constant.

Lemma 0.6. $s \in O(\log |E|) \subset O(|E|)$.

Proof. Obvious. □

Lemma 0.7. $s \in O(k \log \log k) \subset O(k^2)$.

Proof. Since $\log$ is analytic on its domain, by Taylor's Theorem,

$$\begin{aligned} \log(x - \epsilon) &= \log x + \sum_{i=1}^{\infty} \frac{1}{i!} (-\epsilon)^i \frac{d^i}{dx^i} \log x \\ &= \log x + \sum_{i=1}^{\infty} \frac{1}{i!} \epsilon^i (-1)^i \frac{(-1)^{i-1} (i-1)!}{x^i} \\ &= \log x + \sum_{i=1}^{\infty} \frac{-\epsilon^i}{ix^i} \\ &\leq \log x - \frac{\epsilon}{x}, \end{aligned}$$

so $\log k - \log(k - \epsilon) \geq \frac{\epsilon}{k}$ and

$$\begin{aligned} \frac{1}{\log k - \log(k - \epsilon)} &\leq \frac{k}{\epsilon} \\ s &\leq 1 + \frac{k(\log C + \log |E| + \log(\log k + 1))}{\epsilon}, \end{aligned} \quad (8)$$

so $s \in O(k \log \log k)$. □

Lemma 0.8. $s \in O\left(\frac{1}{\epsilon}\right)$.

Proof. Obvious from (8). □

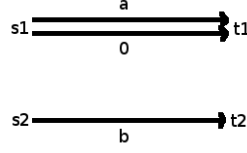


Figure 1: Under certain conditions described in part B of problem 4, this network design problem allows player 1 to decrease his cost by an infinite factor from the solution found by the proposed algorithm.

- (b) Consider the graph depicted in Figure 1 for a two-player network design problem, where a is arbitrary, $b = \max\left(0, \frac{2a - a\epsilon}{\epsilon}\right)$. Take as an initial guess S that player 1 takes the top link, and player 2 takes the bottom link. The potential $\Phi(S)$ is

$$\Phi(S) = aH_1 + bH_1 = (a + b).$$

Player 2 clearly cannot improve his cost by choosing a different path. Player 1, on the other hand, can switch to the middle path. The resulting candidate solution S' then has potential

$$\Phi(S') = bH_2 = b.$$

We have

$$\begin{aligned} b &> \frac{2a - a\epsilon}{\epsilon} \\ b\epsilon &> 2a - a\epsilon \\ (a + b)\epsilon &> 2a \\ \frac{1}{2}(a + b)\epsilon &> a \\ (a + b)\frac{\epsilon}{k} &> a \\ \frac{\epsilon}{k}\Phi(S) &> a, \end{aligned}$$

so the change in potential a is not large. Thus the proposed algorithm terminates at S .

However by switching to S' player 1 decreases his cost from a to 0, an infinite factor.