# CS6998-3: Problem Set # 2

Due in class, on Monday October 27

## Problem 1

A *Groves mechanism* is a sealed-bid auction where each agent $i$ reports a valuation $\tilde{v}_i$, the mechanism selects an efficient allocation $R$ with respect to $\tilde{v}$, and charges each agent $i$ a payment of

$$q_i = h_i(\tilde{v}_{-i}) - \sum_{j \in N-i} \tilde{v}_j(R_j),$$

where $h_i$ is a function of the other agents' reports. In the following, assume that agent valuations are drawn from the domain of general valuations.

(a) Show that in a Groves mechanism, an agent maximizes its utility by reporting its true valuation.

(b) Show that among all individually-rational Groves mechanisms, the VCG mechanism maximizes the payment of each agent.

## Problem 2

Consider a set of $n$ agents whose valuations $v_1, v_2, \ldots, v_n$ are *single-minded*.

(a) Show that the following *nonlinear, anonymous* (i.e., order 2) prices are competitive equilibrium prices.

$$p(S) = \max_{i \in N} v_i(S).$$

That is, given any efficient allocation $R$, the pair $\langle R, p \rangle$ is a competitive equilibrium.

(b) Exhibit a linear program for the efficient allocation problem such that the dual variables corresponding to the "supply equals demand" constraints are nonlinear, anonymous prices. (You do not need to exhibit the dual, but might need to derive it for yourself to ensure your primal is correct.)

## Problem 3

Consider a model with a set of agents $N$ and a set of items $M$ where the number of items equals the number of agents, say $n$. For each agent $i \in N$ let $v_{ij}$ be the agent's value for item $j \in M$. Each agent has a *unit-demand* valuation, meaning that the value of bundle $S$ to a typical agent $i$ is

$$v_i(S) = \max_{j \in S} v_{ij}.$$

In words, given a bundle $S$, the agent keeps the item it values the most in $S$ and discards the rest (hence "unit-demand"). Assume further that $v_{ij}$ is of the form $v_{ij} = a_i b_j$, where $a_1 > a_2 > \ldots > a_n > 0$ and $b_1 > b_2 > \ldots > b_n > 0$. This means that each agent strictly prefers item 1 to item 2, item 2 to item 3, and so on.

(a) Show that the unique efficient allocation is to give item 1 to agent 1, item 2 to agent 2, and so on.

(b) Let $(p_1, p_2, \ldots, p_n)$ be linear, anonymous (i.e., order 1) prices. These are competitive equilibrium prices if and only if the following hold. (You should convince yourself of this.)

$$\begin{aligned} v_{ii} - p_i &\geq v_{ij} - p_j \qquad \forall i \in N, j \in M \\ p_i &\geq 0 \qquad\qquad\;\; \forall i \in N \end{aligned}$$

Show that this system of inequalities can be simplified to the following.

$$\begin{aligned} v_{ii} - p_i &\geq v_{ii+1} - p_{i+1} \qquad \text{for } i = 1, \ldots, n-1 \\ v_{ii} - p_i &\geq v_{ii-1} - p_{i-1} \qquad \text{for } i = 2, \ldots, n \\ p_n &\geq 0 \end{aligned}$$

(c) Show that if $p$ and $p'$ are order 1 competitive equilibrium prices, then so are prices $p \wedge p'$. The latter is the price vector whose $i$th component is $\min\{p_i, p'_i\}$.

(d) The previous part implies that there is a unique *minimal* competitive equilibrium price vector $\bar{p}$, such that $p_j \geq \bar{p}_j$ (for all $j \in M$) for any other linear, anonymous competitive equilibrium prices $p$. Verify that

$$\bar{p}_i = \sum_{j>i} a_j (b_{j-1} - b_j).$$

(Note that this construction proves that linear, anonymous competitive equilibrium prices always exist.)

(e) What is the VCG payment of agent $i$, in terms of the $a_i$'s and $b_j$'s?