CS6998-3: Problem Set # 1

Due in class, on Monday October 13

Problem 1

Recall in class we proved a bound of $\frac{4}{3}$ on the price of anarchy in networks with linear latency functions. We also stated (without proof) that the two node, two link networks form the worst case for price of anarchy calculations.

(a) For a positive integer $d \geq 1$, let $\mathcal{M}_d$ denote the set $\{ax^i : a \geq 0, i \in \{0, 1, \ldots, d\}\}$ of monomials with non negative coefficients with degree at most d. What is the largest price of anarchy possible with cost functions in $\mathcal{M}_d$? Can you give an asymptotic bound in terms of d.

(b) For a positive integer $d \geq 1$, let $\mathcal{P}_d$ denote the set $\{\sum_{i=0}^{d} a_i x^i : a_0, a_1, \ldots, a_d \geq 0\}$ of polynomials with non negative coefficients and degree at most d. What is the largest price of anarchy possible with cost functions in $\mathcal{P}_d$?

Problem 2

When talking about selfish routing in class, we focused on what are known as "single-commodity" networks. That is all traffic has the same source, and same destination. In this problem we extend the definitions to a "multicommodity" networks. We are given a graph $G = (V, E)$ and linear latency functions $\ell_e : [0, 1] \to \mathbb{R}^+$ on all of the edges. We are given k source-sink pairs $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$ with $s_i, t_i \in V$. We would like to route 1 unit of flow from each source s_i to each sink t_i.

(a) Extend the definitions of flows at Nash equilibrium and optimal flows to the multicommodity case.

(b) Prove a generalization of the bicriteria theorem we showed in class. That is, if f is a Nash flow, then $f/2$ is the optimal flow for routing 1/2 unit from each s_i to each t_i.

Problem 3

In class we showed that the price of anarchy for the network design game can be as bad as k. Prove that the price of anarchy can never exceed k.

Problem 4

Recall the network design game as we talked about in class. We are given a graph $G = (V, E)$, and k players, where player i wants to go from $s_i \in V$ to $t_i \in V$. Assume the edge costs are integral, and bounded above by C; that is $c_e \in \{0, 1, 2, \ldots, C-1, C\}$. As we saw, iterated best response will converge to a Nash equilibrium, but may do so slowly. One way to deal with this problem is to consider only "large" changes in the potential. That is, fix a constant $\epsilon > 0$, and call a change in the potential *large* if it decreases Φ by at least an ϵ/k *fraction* of its value.

(a) Show that if we consider only best responses that cause large changes in the potential, the process converges in time that is polynomial in $|E|$, k, $\log C$ and $\frac{1}{\epsilon}$.

(b) Show that the resulting outcome may be far from an equilibrium. That is, even though there are no deviations that cause large changes in the potential, individual players may still reduce their costs by an arbitrarily large factor.