

Biometrics: Fall 2013

Assignment 2: Problems from the textbook: Pattern Classification

Problem 2.12

Let $\omega_{max}(\mathbf{x})$ be the state of nature for which $P(\omega_{max}|\mathbf{x}) \geq P(\omega_i|\mathbf{x})$ for all $i, i = 1, \dots, c$.

- (a) Show that $P(\omega_{max}|\mathbf{x}) \geq 1/c$.
- (b) Show that for the minimum-error-rate decision rule the average probability of error is given by

$$P(error) = 1 - \int P(\omega_{max}|\mathbf{x})p(\mathbf{x})d\mathbf{x}.$$

- (c) Use these two results to show that $P(error) \leq (c-1)/c$.
- (d) Describe a situation for which $P(error) = (c-1)/c$.

Problem 2.13

In many pattern classification problems one has the option either to assign the pattern to one of c classes, or to *reject* it as being unrecognizable. If the cost for rejects is not too high, rejection may be a desirable action. Let

$$\lambda(\alpha_i|\omega_j) = \begin{cases} 0 & i = j \\ \lambda_r & i = c+1 \\ \lambda_s & \text{otherwise,} \end{cases} \quad i, j = 1, \dots, c$$

where λ_r is the loss incurred for choosing the $(c+1)$ th action, rejection, and λ_s is the loss incurred for making any substitution error. Show that the minimum risk is obtained if we decide ω_i if $P(\omega_i|\mathbf{x}) \geq P(\omega_j|\mathbf{x})$ for all j and if $P(\omega_i|\mathbf{x}) \geq 1 - \lambda_r/\lambda_s$, and reject otherwise. What happens if $\lambda_r = 0$? What happens if $\lambda_r > \lambda_s$?

Problem 2.14 (6000 level)

Consider the classification problem with rejection option.

- (a) Use the results of Problem 2.13 to show that the following discriminant functions are optimal for such problems:

$$g_i(\mathbf{x}) = \begin{cases} p(\mathbf{x}|\omega_i)P(\omega_i) & i = 1, \dots, c \\ \frac{\lambda_s - \lambda_r}{\lambda_s} \sum_{j=1}^c p(\mathbf{x}|\omega_j)P(\omega_j) & i = c + 1. \end{cases}$$

- (b) Plot these discriminant functions and the decision regions for the two-category one-dimensional case having

- $p(x|\omega_1) \sim N(1, 1)$
- $p(x|\omega_2) \sim N(-1, 1)$
- $P(\omega_1) = P(\omega_2) = 1/2$, and
- $\lambda_r/\lambda_s = 1/4$.

- (c) Describe qualitatively what happens as λ_r/λ_s is increased from 0 to 1.

Problem 2.23 (6000 level)

Consider the three-dimensional normal distribution $p(\mathbf{x}|\omega) \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where

$$\boldsymbol{\mu} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \text{ and } \boldsymbol{\Sigma} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 2 \\ 0 & 2 & 5 \end{pmatrix}.$$

- (a) Find the probability density at the point $\mathbf{x}_0 = (.5, 0, 1)^t$.
- (b) Construct the whitening transformation $\mathbf{A}_w$ (Eq. 44). Compute the matrices representing eigenvectors and eigenvalues, $\boldsymbol{\Phi}$ and $\boldsymbol{\Lambda}$. Next, convert the distribution to one centered on the origin with covariance matrix equal to the identity matrix, $p(\mathbf{x}|\omega) \sim N(\mathbf{0}, \mathbf{I})$.

Eq.44 in the book : $\mathbf{A}_w = \boldsymbol{\Phi}\boldsymbol{\Lambda}^{-1/2}$.

- (c) Apply the same overall transformation to $\mathbf{x}_0$ to yield a transformed point $\mathbf{x}_w$.
- (d) By explicit calculation, confirm that the Mahalanobis distance from $\mathbf{x}_0$ to the mean $\boldsymbol{\mu}$ in the original distribution is the same as for $\mathbf{x}_w$ to $\mathbf{0}$ in the transformed distribution.

Problem 2.27

Suppose we have two normal distributions with the same covariances but different means: $N(\boldsymbol{\mu}_1, \boldsymbol{\Sigma})$ and $N(\boldsymbol{\mu}_2, \boldsymbol{\Sigma})$. In terms of their prior probabilities $P(\omega_1)$ and $P(\omega_2)$, state the condition that the Bayes decision boundary *not* pass between the two means.